# A Distribution-Free Measure of the Significance of CER Fit Parameters Established Using GERM

## Presented to: 2nd Annual Meeting of the Department of the Navy Cost Analysis Symposium


Presented by

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# Introduction

- General error regression method (GERM) is becoming more and more popular
  - Non-linear CERs using constrained optimization
  - Wide variety of functional forms
  - But, to date, lacks means of evaluating “significance” of individual regression fit parameters
- This research attempts to develop a collection of “significance” metrics for use with GERM
  - Independent of underlying error distribution
  - Comparable across functional forms
  - Require no distributional assumptions
- These “significance” metrics will be beneficial to CER developers
  - Will enable cost modelers to judge “significance” of independent variables
  - Will minimize need to collect unimportant data



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# Background

# Ordinary Least Squares (OLS)

- OLS regression dates back to 1795
  - Carl Friedrich Gauss
- Used most frequently to establish an estimate of the linear relationship between variables *Y* and *X* when the *actual*, though unknown, relationship is assumed to be:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_n X_n + \varepsilon$$

- Where
  - $\beta_i$ are the actual coefficients and
  - $\varepsilon$ is a random error term with $\mu = 0$ and constant $\sigma^2$

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# The Estimated Equation

- The typical result of an OLS regression is an estimate of the relationship, having the form:

$$\hat{y} = b_0 + b_1 x_1 + b_2 x_2 + \cdots + b_n x_n$$

- Where
  - $b_i$ are the estimated coefficients and are calculated by solving the following matrix equation for **b**:

$$\mathbf{b} = \left(\mathbf{X}^{\mathbf{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\mathbf{Y}$$

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# OLS "Significance"

- Modern statistical applications often require tests of the "significance" of the coefficients, $b_i$
- Assuming the error term of the *actual* relationship, $\varepsilon$, has a *normal* distribution, statistical theory shows that the *Student's t* distribution (which is derived from the *normal* distribution) should be used to test significance hypotheses regarding these coefficients
- In statistics, a result is deemed "statistically significant" if that result is *not likely* to have occurred by chance
- Therefore, in OLS we test whether or not each coefficient is really just zero, given that a non-zero estimate was produced



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# The *t*-Statistic

- In OLS, each regression coefficient is scored using the *t*-statistic
- The *t*-statistic is the solution to the test statistic obtained by performing the following hypothesis test:

$$H_o : \beta_i = 0$$

$$H_a : \beta_i \neq 0$$

- The test statistic is:

$$t_{b_i} = \frac{b_i - \beta_i}{SE / \sqrt{\sum (x_i - \bar{X})^2}}$$

- Where
  - $b_i$ is the estimated value of the coefficient
  - $\beta_i$ is the true value of the coefficient (assumed under the null hypothesis to be zero)
  - *SE* is the standard error of the regression



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# The *t*-Test

- The test statistic is then compared to a *Student's t* distribution with *n* degrees of freedom, where *n* is the number of data points minus the number of coefficients
- If the test statistic falls within a critical region, determined in advance, of the tails of the *Student's t* distribution, then the null hypothesis is rejected, and the coefficient $b_i$ is said to be "statistically significant"
- This means that the non-zero value of $b_i$ is unlikely to have been arrived at by chance and therefore $\beta_i$ probably should not be considered as zero in the regression relationship

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# Significant vs. Insignificant

- In practical terms, those coefficients that are “significant” imply that the corresponding cost drivers have something important to say about the value of *Y*
- Those that are NOT “significant” are probably due merely to chance, and thus have little of importance to say about the value of *Y*
- The usual response, then, is to one-by-one remove an “insignificant” variable from the cost model, establish a new regression equation without that variable, then examine the remaining coefficients for significance
- The process is repeated until all remaining variables are deemed “significant”



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# General Error Regression Method

- GERM as often practiced today dates back to work done during the period 1994-1998 by S.A. Book, P.H. Young, and N.Y. Lao
- GERM refers to the regression method in which estimates of the fit parameters of generalized functional forms (e.g. non-linear) are derived through constrained optimization
  - NOTE: GERM as discussed here is NOT the same as Iteratively Re-weighted Least Squares (IRLS)
- GERM enables derivation of a regression model regardless of functional form or nature of error distribution
  - Seeks fit parameters that minimize SSE or SSPE
- Two common varieties of GERM models:
  - Those with ADDITIVE errors
  - Those with MULTIPLICATIVE errors

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# No Analogous "Significance" Test for GERM Exists

- Until now there has been no "significance" test for fit parameters derived using GERM that is analogous to the *t*-statistic found in OLS
  - GERM CERs typically non-linear
  - Error distributions not usually assumed to be normal
- CER developers usually determine, in advance, which independent variables they want in their models
  - Then judge goodness of fit of entire CER based on *SE* or *SPE* and Pearson's $r^2$ only
  - If a CER can be developed with low *SE* or *SPE* and high Pearson's $r^2$ then the CER is considered a success
- But, it would be beneficial to be able to judge whether or not one or more of the "desired" independent variables (cost drivers) actually have something important to say about the value of the dependent variable (cost)



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# What is Needed

- Goal of this research is to "invent" a collection of metrics that
  - Are analogous to the t-statistic
  - Are comparable among CERs regardless of functional form
  - Require no distributional assumptions
  - Simple to understand
  - Enable one to judge "significance" of regression fit parameters
- Consider a CER of the form $Y = a + bX^c W^d Q^e f^{Type}$
- If any of the fit parameters do not substantially impact the calculated value of the CER, or reduce the model's variance, then they might as well be removed from the model – one less data point to collect, one more degree of freedom



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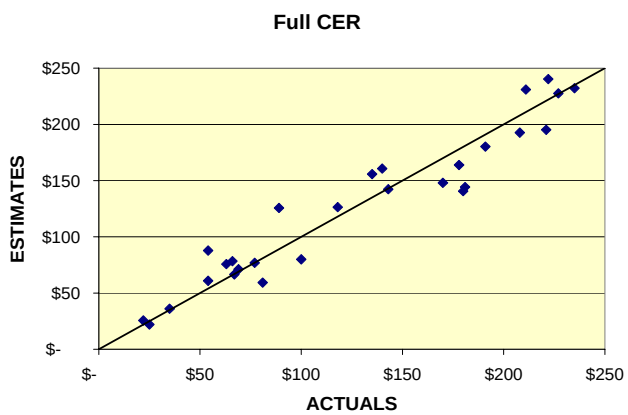
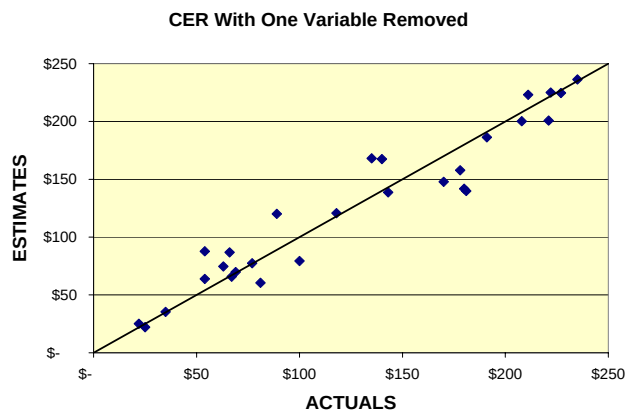
# “Insignificant” Fit Parameters

- Hypothesis: “Insignificant” fit parameters will have little impact on CER result or variance if nullified
- Consider the two estimates vs. actuals plots shown below
  - The one on the left is based on a multivariable GERM CER
  - The one on the right is based on a re-optimized GERM CER after one of the fit parameters is nullified




- Note that they are nearly identical!
  - Indicates that the nullified fit parameter didn’t make much difference – so, should be considered “insignificant”

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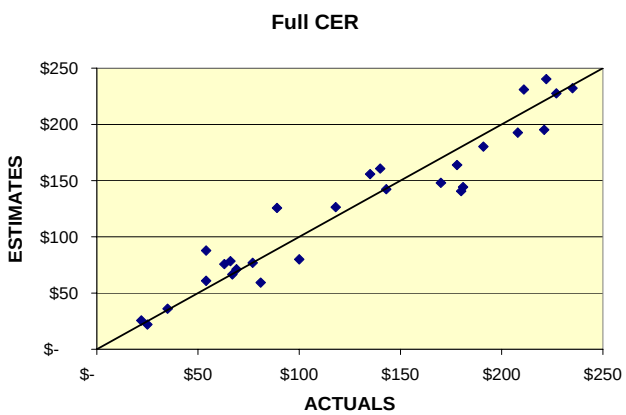
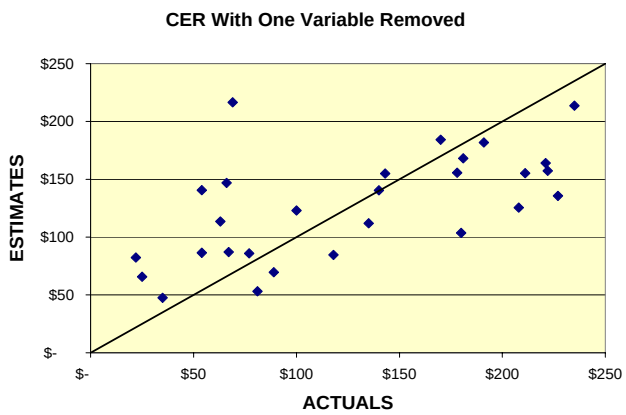
# “Significant” Fit Parameters

- Hypothesis: “Significant” fit parameters will have substantial impact on CER result or variance if nullified
- Consider the two estimates vs. actuals plots shown below
  - The one on the left is based on a multivariable GERM CER
  - The one on the right is based on a re-optimized GERM CER after one of the fit parameters is nullified




- This time they are quite different!
  - Indicates that the nullified fit parameter made a big difference – so, should be considered “significant”

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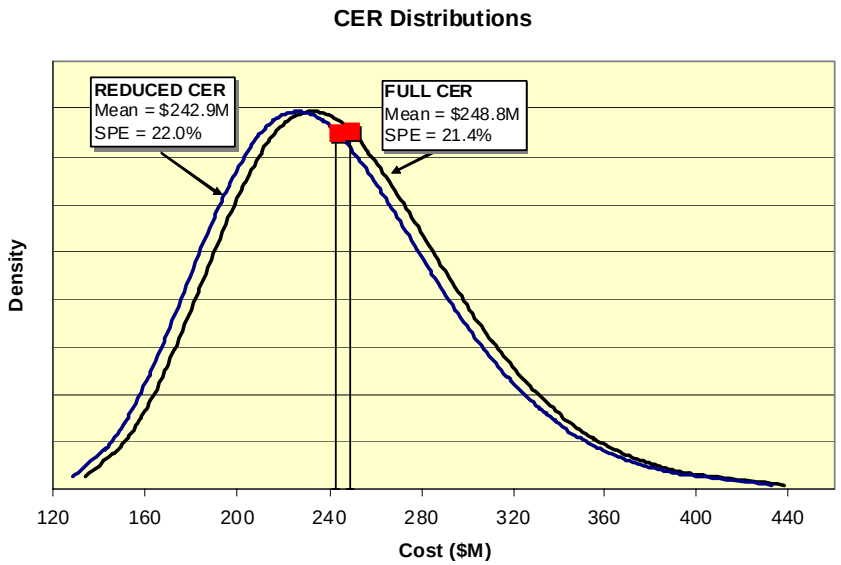
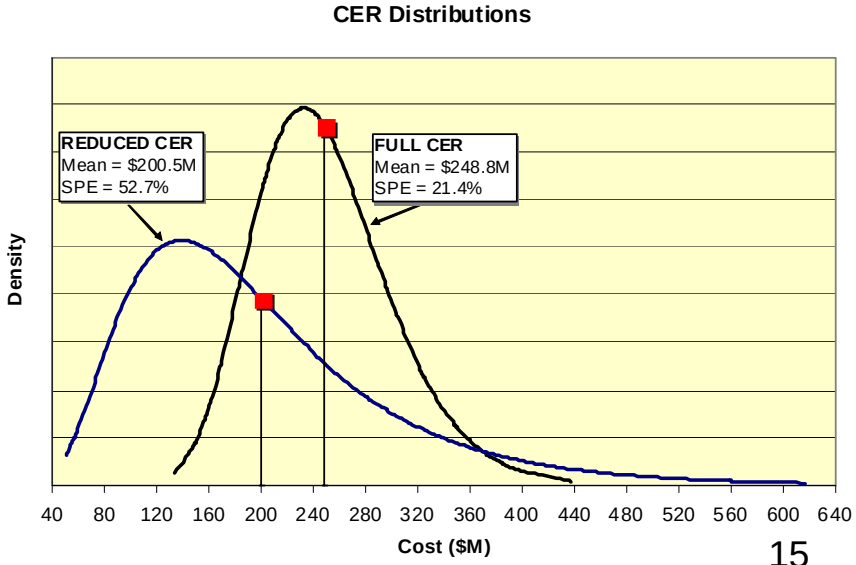
# Impact on CER Mean and Variance

- If an “insignificant” fit parameter is nullified, the resulting error distribution’s mean and variance should remain about the same
  - The error distributions on the left indicate nullification of an “insignificant” fit parameter
- If a “significant” fit parameter is nullified, the resulting error distribution’s mean and variance should change dramatically
  - The error distributions on the right indicate nullification of a “significant” fit parameter

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# Proposition: Fit Parameters Can Be Scored Based on Impact to CER Mean and Variance

- It is proposed that individual CER fit parameters can be scored by measuring changes in CER mean and variance when those fit parameters are nullified
- Scoring metric should produce a small number if the change in mean and/or variance is small (insignificant), and a large number if the change in mean and/or variance is large (significant)
- Assumption: The CER has a probability distribution
  - We may not know what it looks like
  - But it should have a mean and a variance
  - GERM CERs easily meet this assumption

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# A Note on "Nullification"

- How, exactly, does one "nullify" a fit parameter?
- There is an element of art to this, and it depends on the functional form
- We desire a method to make the fit parameter "go away"
- Consider the CER: $Y = a + bX^{c}W^{d}Q^{e}f^{Type}$
- The method used to nullify each fit parameter depends on its position in the CER
- Fit parameter *a* is an additive constant
  - To nullify, set it equal to zero
- Fit parameters *b* and *f* are multipliers
  - To nullify, set them equal to 1.0
- Fit parameters *c*, *d*, and *e* are exponents
  - To nullify, set them equal to zero

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# Significance Relative to the Mean, $SIG_{Mean}$

- The first significance metric is that relative to the CER mean
- Define the "pseudo-mean" of a CER, *f(X),* as

$$f_{\bar{Y}}(X) = f\left(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_N\right)$$

- $SIG_{Mean}$ is defined as the percentage difference between the pseudo-mean of the ***full*** CER and the pseudo-mean of the ***reduced*** CER

$$SIG_{Mean} = \frac{f_{\bar{Y}}(X)_{\text{Reduced}} - f_{\bar{Y}}(X)_{\text{Full}}}{f_{\bar{Y}}(X)_{\text{Full}}}$$

- The ***full*** CER contains all of the CER's fit parameters
- In the ***reduced*** CER, one of the fit parameters is nullified
- Both CERs are evaluated at the means of the independent variables (cost driver values)

Note: The "pseudo-mean" of a CER is the same
As the mean only if the CER is linear

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# $SIG_{Mean}$ Example #1 Setup

- Consider the following ***full*** CER:

$$f(X) = -79.65 + 31.35(X_1)^{0.3664}(X_2)^{0.1094}(X_3)^{0.0576}(1.44)^{X_4}$$

- The pseudo-mean is:

$$f_{\bar{Y}}(X)_{\text{Full}} = -79.65 + 31.35(\bar{X}_1)^{0.3664}(\bar{X}_2)^{0.1094}(\bar{X}_3)^{0.0576}(1.44)^{\bar{X}_4}$$

- Suppose:

$$\bar{X}_1 = 108,\ \bar{X}_2 = 12,\ \bar{X}_3 = 9.9,\ \bar{X}_4 = 0.6$$

- Then the evaluated pseudo-mean is:

$$f_{\bar{Y}}(X)_{\text{Full}} = -79.65 + 31.35(108)^{0.3664}(12)^{0.1094}(9.9)^{0.0576}(1.44)^{0.6}$$
$$= \$245.22$$

- Now suppose we nullify the exponent associated with $X_1$, by setting it equal to zero



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# $SIG_{Mean}$ Example #1 Conclusion

- After re-optimizing, we are left with the following ***reduced*** CER:

$$f_{\bar{Y}}(X)_{\text{Reduced}} = 151.96 + 9.03(\bar{X}_1)^0(\bar{X}_2)^{-0.3739}(\bar{X}_3)^{0.5594}(3.60)^{\bar{X}_4}$$

- i.e.:

$$f_{\bar{Y}}(X)_{\text{Reduced}} = 151.96 + 9.03(\bar{X}_2)^{-0.3739}(\bar{X}_3)^{0.5594}(3.60)^{\bar{X}_4}$$

- Re-entering the values for $\bar{X}_2, \bar{X}_3, and\ \bar{X}_4$ we have:

$$f_{\bar{Y}}(X)_{\text{Reduced}} = 151.96 + 9.03(12)^{-0.3739}(9.9)^{0.5594}(3.60)^{0.6} = \$179.69$$

- The value calculated for $SIG_{Mean}$ is:

$$SIG_{Mean} = \frac{179.69 - 245.22}{245.22} = -0.267 = -26.7\%$$

- In this case, the difference is quite large, indicating that the fit parameter for $X_1$ is ***"significant"***



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# $SIG_{Mean}$ Example #2

- Now consider the same ***full*** CER:

$$f(X) = -79.65 + 31.35(X_1)^{0.3664}(X_2)^{0.1094}(X_3)^{0.0576}(1.44)^{X_4}$$

- But this time we nullify the exponent associated with $X_3$ by setting it equal to zero
- After re-optimizing, we are left with:

$$f_{\bar{Y}}(X)_{\text{Reduced}} = -63.10 + 29.23(\bar{X}_1)^{0.3962}(\bar{X}_2)^{0.0924}(1.485)^{\bar{X}_4}$$

- The pseudo-mean is evaluated as:

$$f_{\bar{Y}}(X)_{\text{Reduced}} = -63.10 + 29.23(108)^{0.3962}(12)^{0.0924}(1.485)^{0.6} = \$234.91$$

- Then the value calculated for $SIG_{Mean}$ is:

$$SIG_{Mean} = \frac{234.91 - 245.22}{245.22} = -0.042 = -4.2\%$$

- This time, the mean of the ***reduced*** CER is only about 4% less than that of the ***full*** CER – indicating that this fit parameter is relatively ***"insignificant"***



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# Significance Relative to the Standard Error, $SIG_{SE}$

- For CERs with additive errors, the second significance metric is that relative to the standard error (*SE*) of the CER, denoted $SIG_{SE}$
- The *SE* of the CER distribution is computed as follows:

$$SE = \sqrt{\frac{1}{n-m}\sum_{i=1}^{n}\left[y_i - f(x_i)\right]^2}$$

- $SIG_{SE}$ is defined as the percentage difference between the *SE* of the ***full*** CER and *SE* of the ***reduced*** CER

$$SIG_{SE} = \frac{SE_{\text{Reduced}} - SE_{\text{Full}}}{SE_{\text{Full}}}$$

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# $SIG_{SE}$ Example #1 Setup

- Consider the following ***full*** CER (with additive errors):

$$f(X)_{\text{Full}} = 11.41 + 5.38(X_1)^{0.6115}(X_2)^{0.1487}(X_3)^{0.0793}(1.71)^{X_4}$$

- The *SE* of the ***full*** CER is calculated as:

$$SE_{\text{Full}} = \sqrt{\frac{1}{n-m}\sum_{i=1}^{n}\left[y_i - f(x_i)_{\text{Full}}\right]^2} = \$40.84$$

- Now suppose we nullify the exponent associated $X_1$, by setting it equal to zero
- After re-optimizing, we are left with the following ***reduced*** CER:

$$f(X)_{\text{Reduced}} = 165.47 + 2.68 \times 10^{-6}(X_1)^0(X_2)^{1.1959}(X_3)^{4.3383}(18.37)^{X_4}$$

- i.e.:

$$f(X)_{\text{Reduced}} = 165.47 + 2.68 \times 10^{-6}(X_2)^{1.1959}(X_3)^{4.3383}(18.37)^{X_4}$$



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# $SIG_{SE}$ Example #1 Conclusion

- The *SE* of the ***reduced*** CER is calculated, after re-optimization, as:

$$SE_{\text{Reduced}} = \sqrt{\frac{1}{n-m}\sum_{i=1}^{n}\left[y_i - f(x_i)_{\text{Reduced}}\right]^2} = \$95.06$$

- Then the value of $SIG_{SE}$ is calculated as:

$$SIG_{SE} = \frac{\$95.06 - \$40.84}{\$40.84} = 1.328 = 132.8\%$$

- In this case, the difference between the *SE* of the ***full*** and ***reduced*** CERs is quite large
  - Indicating that the fit parameter for the exponent associated with $X_1$ is ***"significant"***



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# $SIG_{SE}$ Example #2

- Now consider the same ***full*** CER, but this time nullify the exponent associated with $X_3$ by setting it equal to zero
- After re-optimizing, we are left with the following ***reduced*** CER:

$$f(X)_{\text{Reduced}} = 15.35 + 5.52(X_1)^{0.6395}(X_2)^{0.1412}(1.71)^{X_4}$$

- The *SE* of the ***reduced*** CER is calculated as:

$$SE_{\text{Reduced}} = \sqrt{\frac{1}{n-m}\sum_{i=1}^{n}\left[y_i - f(x_i)_{\text{Reduced}}\right]^2} = \$44.04$$

- And the value of $SIG_{SE}$ is calculated as:

$$SIG_{SE} = \frac{\$44.04 - \$40.84}{\$40.84} = 0.078 = 7.8\%$$

- In this case $SIG_{SE}$ is much smaller, indicating that the fit parameter is relatively ***"insignificant"***



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# Significance Relative to the Standard Percent Error, $SIG_{SPE}$

- For CERs with multiplicative errors, the third significance metric is that relative to the standard percent error (*SPE*) of the CER, denoted $SIG_{SPE}$
- The *SPE* of the CER distribution is computed as follows:

$$SPE = \sqrt{\frac{1}{n-m}\sum_{i=1}^{n}\left[\frac{f(x_i)-y_i}{f(x_i)}\right]^2}$$

- $SIG_{SPE}$ is defined as the percentage difference between the *SPE* of the ***full*** CER and *SPE* of the ***reduced*** CER

$$SIG_{SPE} = \frac{SPE_{\text{Reduced}} - SPE_{\text{Full}}}{SPE_{\text{Full}}}$$

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# $SIG_{SPE}$ Example #1 Setup

- Consider the following ***full*** CER (multiplicative errors):

$$f(X)_{\text{Full}} = -79.65 + 31.35(X_1)^{0.3664}(X_2)^{0.1094}(X_3)^{0.0576}(1.44)^{X_4}$$

- The *SPE* of the ***full*** CER is calculated as:

$$SPE_{\text{Full}} = \sqrt{\frac{1}{n-m}\sum_{i=1}^{n}\left[\frac{f(x_i)_{\text{Full}} - y_i}{f(x_i)_{\text{Full}}}\right]^2} = 21.4\%$$

- Now suppose we nullify the exponent associated $X_1$, by setting it equal to zero
- After re-optimizing, we are left with the following ***reduced*** CER:

$$f(X)_{\text{Reduced}} = 151.96 + 9.0322(X_1)^0(X_2)^{-0.3739}(X_3)^{0.5594}(3.60)^{X_4}$$

- i.e.:

$$f(X)_{\text{Reduced}} = 151.96 + 9.0322(X_2)^{-0.3739}(X_3)^{0.5594}(3.60)^{X_4}$$



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# $SIG_{SPE}$ Example #1 Conclusion

- The *SPE* of the ***reduced*** CER is calculated, after re-optimization, as:

$$SPE_{\text{Reduced}} = \sqrt{\frac{1}{n-m}\sum_{i=1}^{n}\left[\frac{f(x_i)_{\text{Reduced}} - y_i}{f(x_i)_{\text{Reduced}}}\right]^2} = 52.7\%$$

- Then the value of $SIG_{SPE}$ is calculated as:

$$SIG_{SPE} = \frac{52.7\% - 21.4\%}{21.4\%} = 1.463 = 146.3\%$$

- In this case, the difference between the *SPE* of the ***full*** and ***reduced*** CERs is quite large
  - Indicating that the fit parameter for the exponent associated with $X_1$ is ***"significant"***



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# $SIG_{SPE}$ Example #2

- Now consider the same ***full*** CER, but this time nullify the exponent associated with $X_3$ by setting it equal to zero
- After re-optimizing, we are left with the following ***reduced*** CER:

$$f(X)_{\text{Reduced}} = -63.10 + 29.23(X_1)^{0.3962}(X_2)^{0.0924}(1.48)^{X_4}$$

- The *SPE* of the ***reduced*** CER is calculated as:

$$SPE_{\text{Reduced}} = \sqrt{\frac{1}{n-m}\sum_{i=1}^{n}\left[\frac{f(x_i)_{\text{Reduced}} - y_i}{f(x_i)_{\text{Reduced}}}\right]^2} = 22.5\%$$

- And the value of $SIG_{SPE}$ is calculated as:

$$SIG_{SPE} = \frac{22.5\% - 21.4\%}{21.4\%} = 0.051 = 5.1\%$$

- In this case $SIG_{SPE}$ is much smaller, indicating that the fit parameter is relatively ***"insignificant"***



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# The Total Significance, $SIG_{Total}$

- The last significance metric proposed here is that for the total significance, which is comprised of a combination of $SIG_{Mean}$ and $SIG_{SE}$ (or $SIG_{SPE}$)
- The simplest approach is to add the absolute values of $SIG_{Mean}$ and $SIG_{SE}$ (or $SIG_{SPE}$), a process that combines the percentage shift in the CER mean with the percentage change in the CER variance
  - $SIG_{Total} = SIG_{SE} + \left|SIG_{Mean}\right|$ (for additive errors)
  - $SIG_{Total} = SIG_{SPE} + \left|SIG_{Mean}\right|$ (for multiplicative errors)
- Note there should be no need to take the absolute value of $SIG_{SE}$ or $SIG_{SPE}$ because they should always be non-negative



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# $SIG_{Total}$ Example

- Suppose we calculate $SIG_{Mean}$ and $SIG_{SE}$ on a certain fit parameter and it turns out that
  - $SIG_{Mean}$ = -0.035 (-3.5%)
  - $SIG_{SE}$ = 0.045 (4.5%)
- Then $SIG_{Total}$ is calculated as follows:
  - $SIG_{Total} = SIG_{SE} + |SIG_{Mean}| = 0.045 + | -0.035 | = 0.080$
  - So the combined significance is 8%
  - This could be considered relatively ***"insignificant"***



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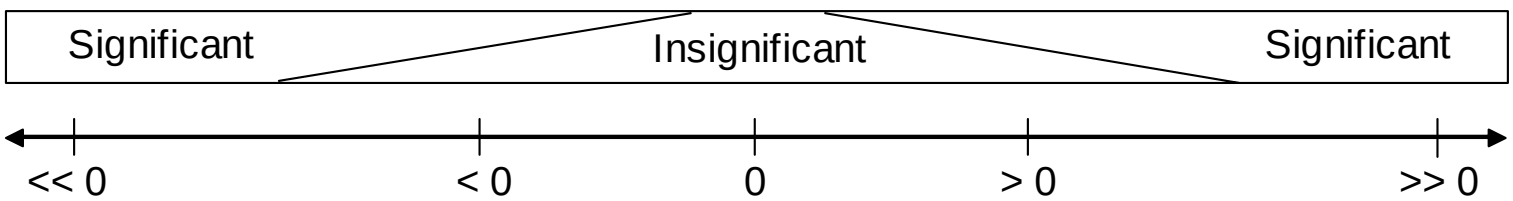
# Significant or Insignificant?

- An obvious question is "at what value of *SIG* is a fit parameter declared ***significant*** or ***insignificant***?"
  - This is still an open question
  - Subject to interpretation
  - More research needed
- For now, the author suggests as default values:
  - $SIG_{SE}$, $SIG_{SPE}$, $SIG_{Mean}$ < 5% indicate ***insignificant***
  - $SIG_{Total}$ < 10% indicates ***insignificant***




- Significance metrics cover a continuum of values
  - Those close to zero are ***insignificant***
  - Those that deviate substantially from zero are ***significant***

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# Conclusions

- This research has demonstrated at least one way of evaluating “significance” of individual fit parameters that are established using GERM
- Metrics are comparable across CERs regardless of functional form
  - They are developed heuristically
  - They require no distributional assumptions – only that the CER error distribution exists and has a finite mean and variance
  - Provide a simple collection of metrics by which to judge the “significance” of individual fit parameters
  - They are beneficial to anyone who uses GERM to develop CERs



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# Recommended Further Study

- This research comprises a “first look” by MCR
- In furtherance of this research, MCR is also looking into:
  - The possible discovery of a “universal” value that determines whether a fit parameter is either significant or insignificant
  - Development of methods to simplify the calculation of each fit parameter’s SIG values, e.g., is there an analytical way to do this?
  - Discovery of more relevant or descriptive metrics
  - Variations on the current theme to include
    - Impact on variance due to changes in degrees of freedom
    - Use on CERs developed with IRLS

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# General Error Regression Method

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  - Seeks fit parameters that minimize SSE or SSPE
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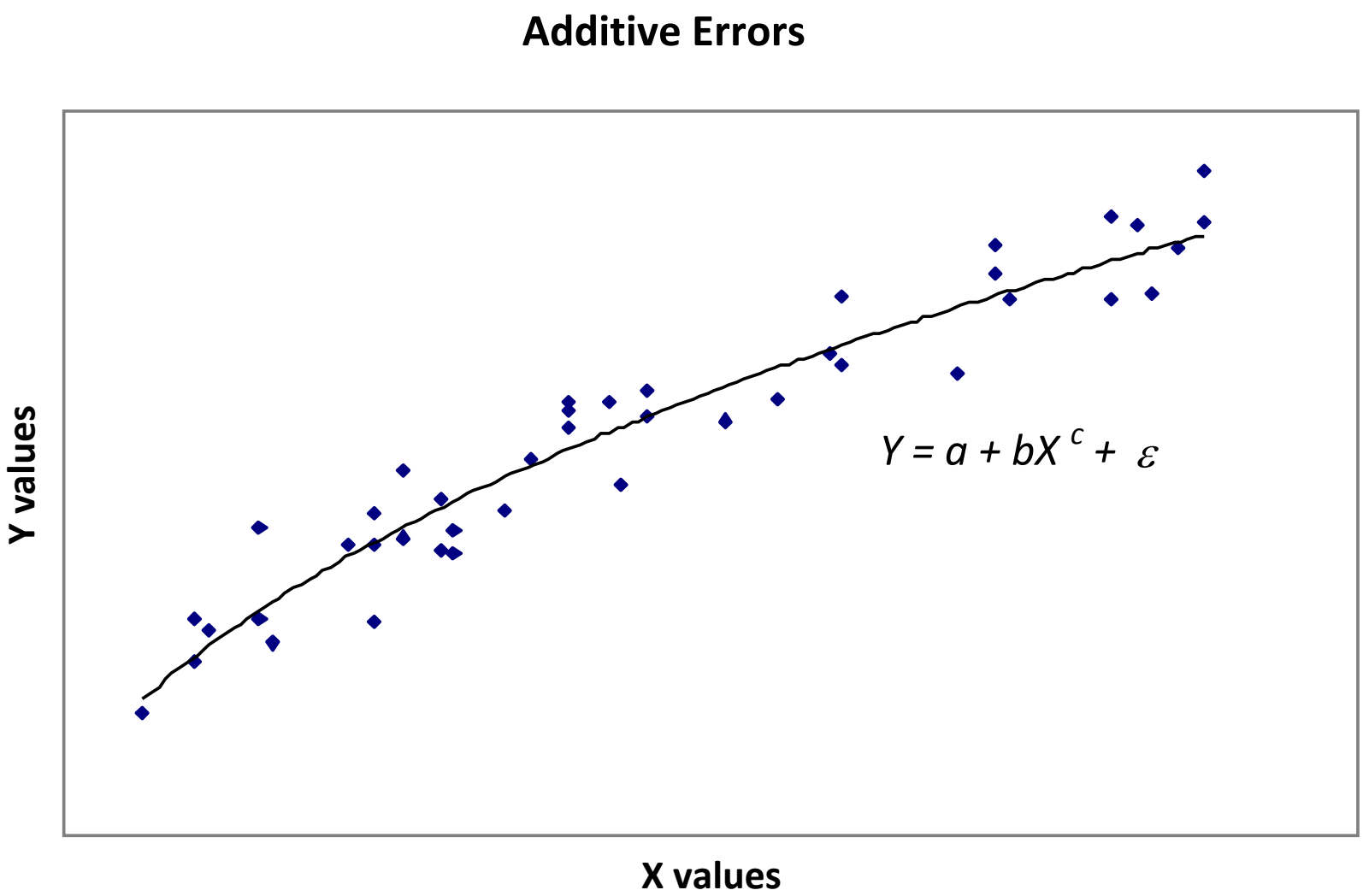
# Additive Errors



# GERM With Additive Errors

- For each $y_i$, the actual value equals the estimated value plus a random error, $\varepsilon$, with $\mu = 0$ and constant $\sigma^2$

$$y_i = f(x_i) + \varepsilon_i$$

- The error is the difference between the actual value, $y_i$, and the estimated value, $f(x_i)$

$$\varepsilon_i = y_i - f(x_i)$$

- The problem is to choose the fit parameters of $f(X)$ so that the sum of squared errors is as small as possible

$$SSE = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left(y_i - f(x_i)\right)^2$$

- Solutions can be found using optimization techniques



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# Additive GERM Statistics

- Average Bias: The average of all errors made in estimating the points in the database
  - This is constrained to be equal to zero in the optimization process

$$Average\ Bias = \frac{1}{n}\sum_{i=1}^{n}\left[f(x_i) - y_i\right] \times 100\%$$

- Standard Error (*SE*): Same interpretation as the *SE* in *OLS* – the root mean square of all errors made in estimating the points in the database

$$SE = \sqrt{\frac{1}{n-m}\sum_{i=1}^{n}\left[y_i - f(x_i)\right]^2}$$

- Pearson's $r^2$: The squared correlation between the estimated values, $f(x_i)$, and the actual values, $y_i$

$$\text{Pearson's } r^2 = \left[\frac{n\sum_{i=1}^{n} y_i f(x_i) - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} f(x_i)}{\sqrt{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}\sqrt{n\sum_{i=1}^{n} f(x_i)^2 - \left(\sum_{i=1}^{n} f(x_i)\right)^2}}\right]$$

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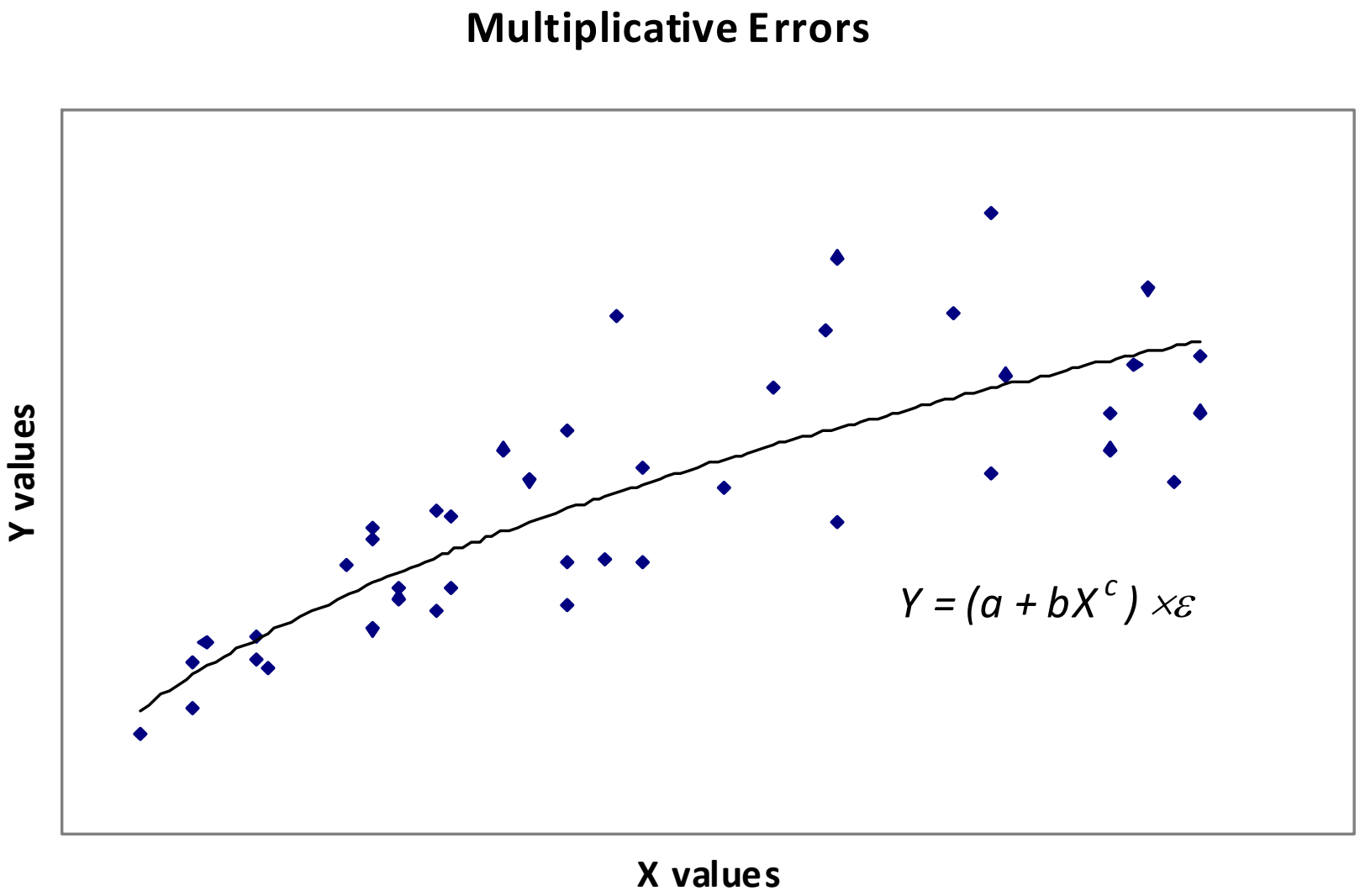
# Multiplicative Errors



# GERM With Multiplicative Errors

- For each $y_i$, the actual value equals the estimated value multiplied by a random error, $\varepsilon$, with $\mu = 1$ and constant $\sigma^2$

$$y_i = f(x_i) \cdot \varepsilon_i$$

- The error is the ratio of the actual value, $y_i$, to the estimated value, $f(x_i)$

$$\varepsilon_i = \frac{y_i}{f(x_i)} = \frac{\text{Actual}}{\text{Estimate}}$$

- The problem is to choose the fit parameters of $f(X)$ so that the summation shown below is as small as possible

$$\sum_{i=1}^{n} (\varepsilon_i - 1)^2 = \sum_{i=1}^{n} \left[ \frac{y_i - f(x_i)}{f(x_i)} \right]^2$$

- Solutions can be found using optimization techniques



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# Multiplicative GERM Statistics

- Average Percent Bias: The average of all percent errors made in estimating the points in the database
  - This is constrained to be equal to zero in the optimization process

$$Average\ Percent\ Bias = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{f(x_i)-y_i}{f(x_i)}\right]\times 100\%$$

- Standard Percent Error (*SPE*): The root mean square of all percent errors made in estimating the points in the database

$$SPE = \sqrt{\frac{1}{n-m}\sum_{i=1}^{n}\left[\frac{f(x_i)-y_i}{f(x_i)}\right]^2}$$

- Pearson's $r^2$: The squared correlation between the estimated values, $f(x_i)$, and the actual values, $y_i$

$$\text{Pearson's } r^2 = \left[\frac{n\sum_{i=1}^{n} y_i f(x_i) - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} f(x_i)}{\sqrt{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}\sqrt{n\sum_{i=1}^{n} f(x_i)^2 - \left(\sum_{i=1}^{n} f(x_i)\right)^2}}\right]$$

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