Why Correlation Matters in Cost Estimating

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Contents

Space-System Cost Elements

0.0	TOTAL BRILLIANT EYES PROGRAM
1.0	SPACE BRILLIANT EYES SYSTEM
1.1	System-Level Costs
1.2	Space Vehicle (SV) Segment
1.2.1	SV Program Level
1.2.2	Space Vehicle Prime Mission Equipment
1.2.2.1	Space Software
1.2.2.2	Space Vehicle
1.2.2.2.1	Space Vehicle IA&T
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1.2.2.2.3	Insertion Vehicle
1.2.2.2.4	Survivability
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1.2.4	Spare Parts
1.2.5	Technology and Producibility
1.2.6	Aerospace Ground Equipment
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1.3	Engineering Change Orders (ECOs)
1.4	Other Government Costs
1.5	Risk

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2.2	Ground Segment
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2.4	Ground Operations and Support
2.5	Ground ECO
2.6	Ground Other Government Costs
2.7	Ground Risk
3.0	LAUNCH BRILLIANT EYES SYSTEM
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3.2.6	Launch Operations
3.3	Launch ECO
3.4	Launch Other Government Costs
3.5	Launch Risk

Contents

- **A Cost Analysis is a Risk Analysis**
- **Measuring Uncertainty in Cost**
- **What is Correlation?**
- **Correlation Impacts Uncertainty**
- **Correlation Also Impacts the Estimate!**
- **Summary**

Typical Cost-Estimating “Roll-Up” Procedure

- List Cost Elements in Work-Breakdown Structure (WBS)
- Calculate “Single Best Estimate” of Cost for Each WBS Element
- Sum All Single Best Estimates
- Define Result to be “Single Best Estimate” of Total-System Cost

What Does “Best” Estimate Signify?

- “Most Likely” Cost? (“Mode”)
- 50th-Percentile Cost? (“Median”)
- Average Cost? (“Mean”)
- These Three are Almost Always Different

Cost-Element Probability Distributions

- Mean, Median, Mode are Statistical Characteristics of Probability Distributions
- Use of These Terms Implicitly Assumes that Costs Have Probability Distributions
- Indeed, Even Admission that “Best” Estimate is not the Only Possible Estimate Implicitly Assumes That Costs Have Probability Distributions

Cost-Risk Analysis

- **Definition of "Cost Risk"**
  - **Inadequacy of Forecasted Funding Requirements to Assure that Program Can Be Completed and Meet Its Stated Objectives**
- **Element Costs Are Uncertain Quantities (i.e., Random Variables) That Have "Probability" Distributions**
  - **Combine Element Cost Distributions to Generate Cumulative Distribution of Total System Cost**
  - **Read off 70th Percentile Cost, 90th Percentile Cost, etc., From Cumulative Distribution to Estimate Amount of Extra Dollars Needed to Cover Risk**
  - **Quantify Confidence in Ability to Complete Program Funded at the "Best" (or any other) Estimate of System Cost**

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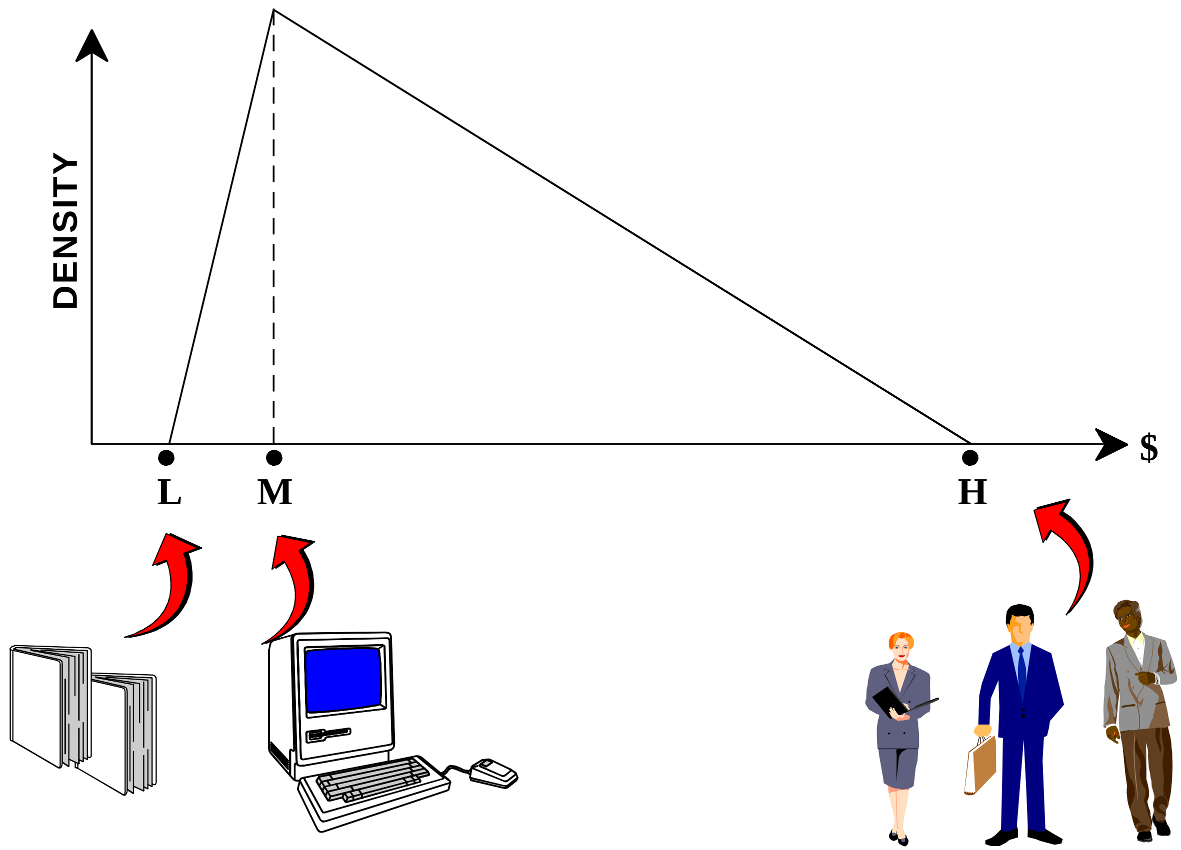
Triangular Distribution of Element Cost

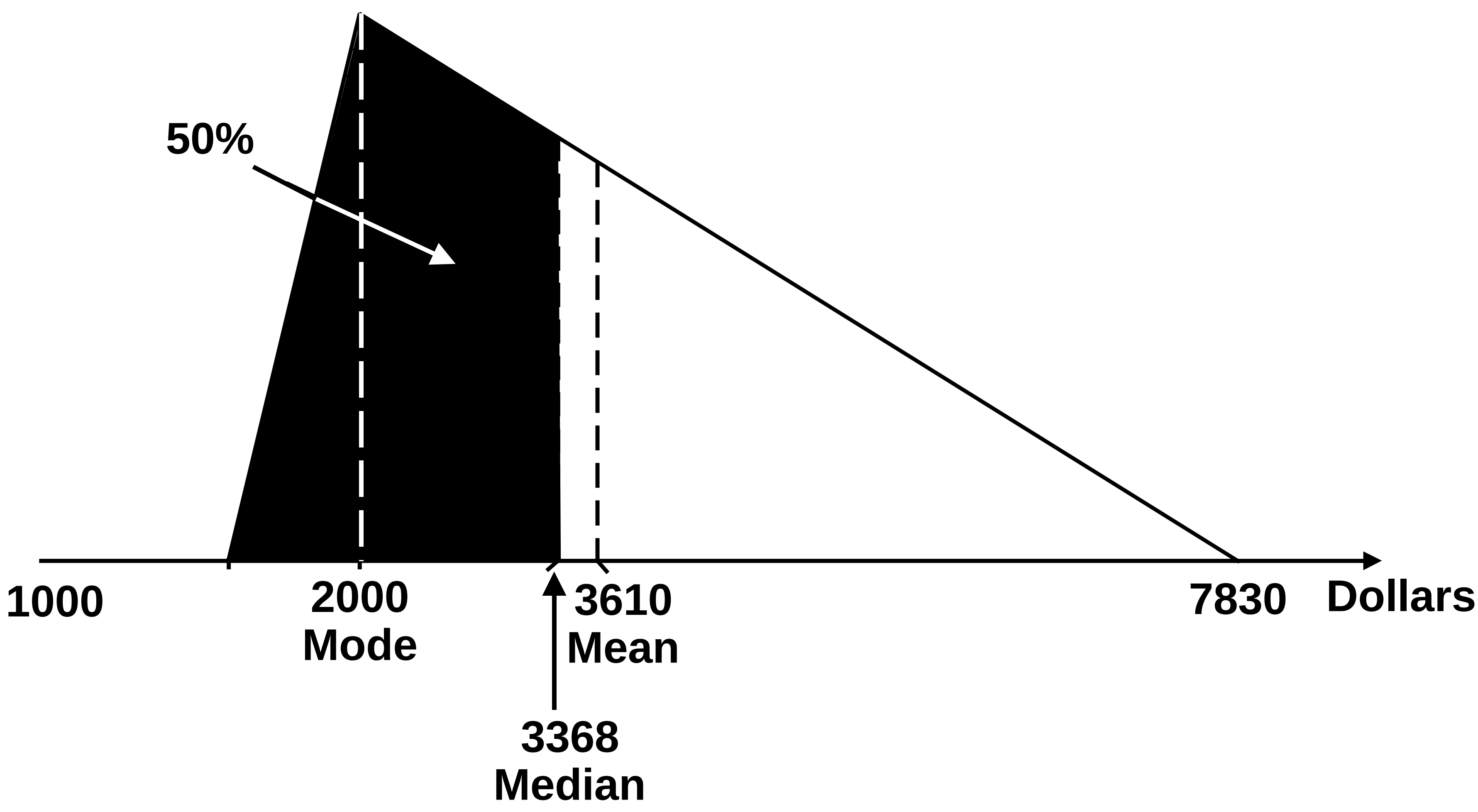


Optimistic Cost

Best-Estimate Cost (Mode)

Cost Implication of Technical Assessment

Triangular Distribution of WBS-Element Cost



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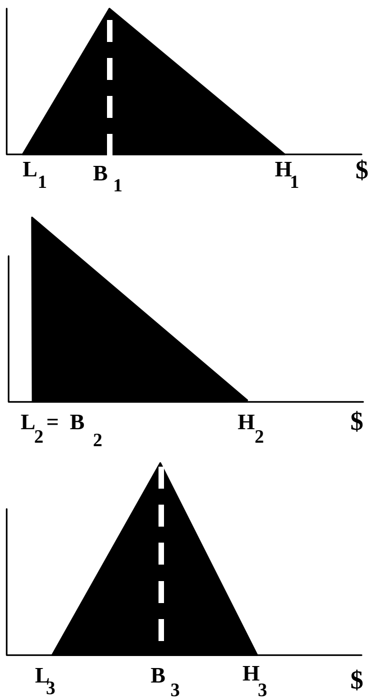
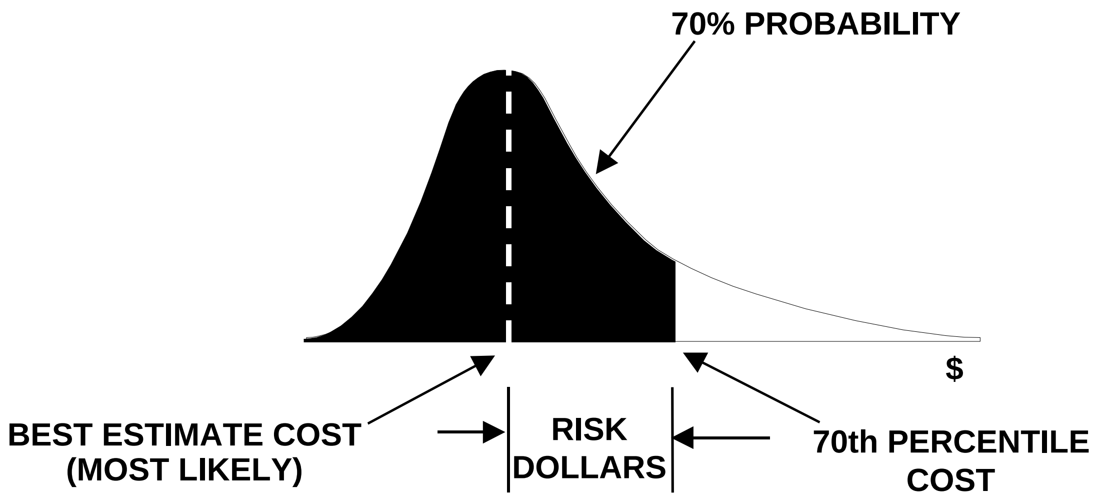
Cost-Risk Procedure

COST-ELEMENT TRIANGULAR DISTRIBUTIONS



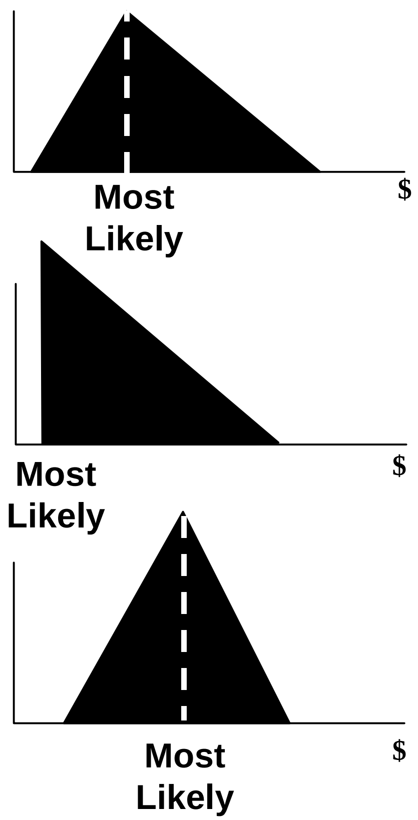
MERGE ELEMENT COST DISTRIBUTIONS INTO TOTAL-COST DISTRIBUTION



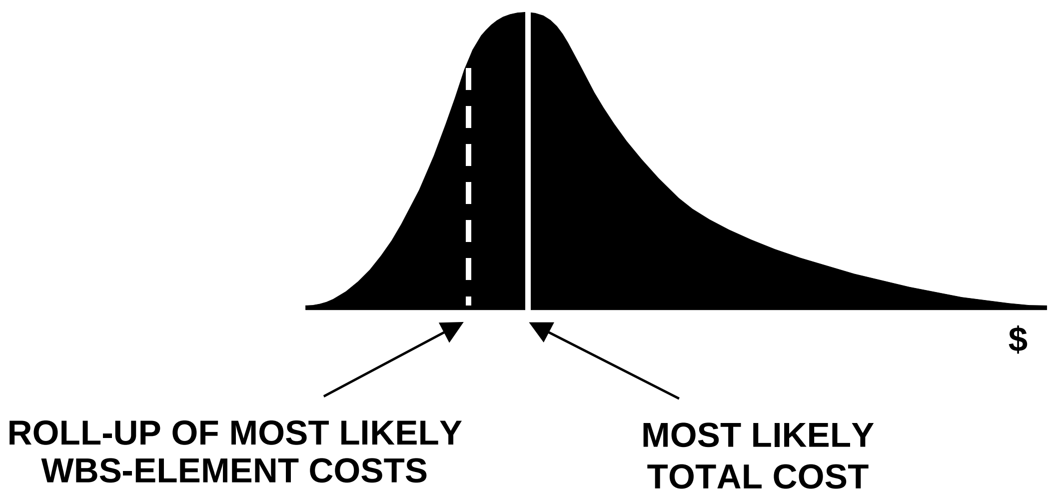
Note: Addition of risk dollars brings confidence that total appropriation (best estimate plus risk dollars) is sufficient to fund program.

When WBS Elements Are Few

WBS-ELEMENT TRIANGULAR COST DISTRIBUTIONS

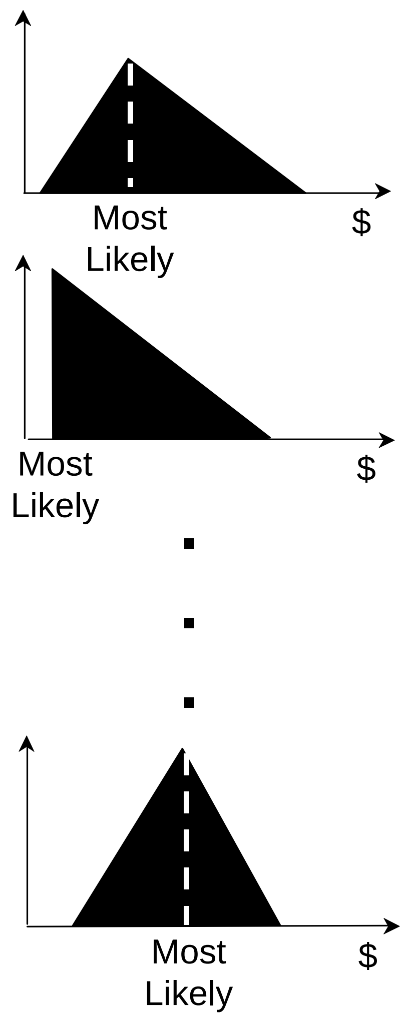


MERGE WBS-ELEMENT COST DISTRIBUTIONS INTO TOTAL-COST LOGNORMAL DISTRIBUTION

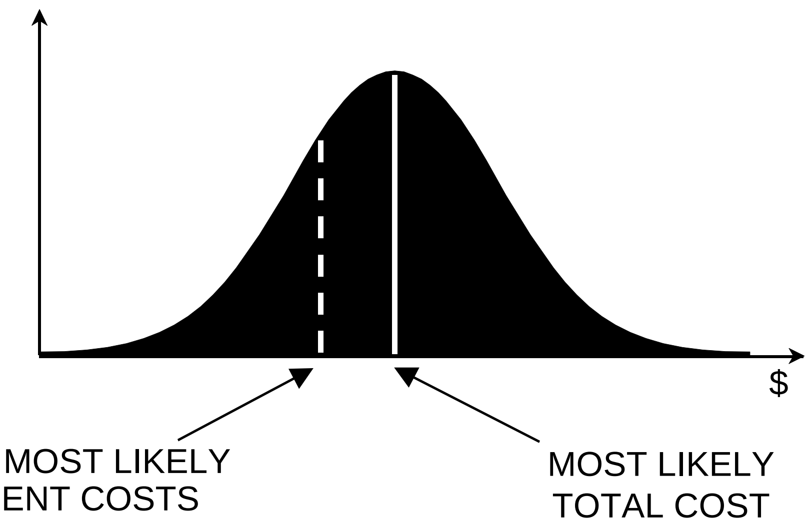


When WBS Elements Are Many

WBS-ELEMENT TRIANGULAR COST DISTRIBUTIONS



MERGE WBS-ELEMENT COST DISTRIBUTIONS INTO TOTAL-COST NORMAL DISTRIBUTION



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Pearson "Product-Moment" Correlation

- **Suppose X and Y are Two Random Variables**
  - $m_X = E(X),\ \ m_Y = E(Y)$ **are their Expected Values ("Means")**
  - **True Theorem:** $E(X+Y) = E(X) + E(Y) = m_X + m_Y$
  - **False Theorem:** $E(XY) = E(X)E(Y)$
  - $Cov(X,Y) = E(XY) - E(X)E(Y) =$ **"Covariance" of X and Y**

- $Var(X) = Cov(X,X) = E\left(X^2\right) - \left[E(X)\right]^2 =$ **"Variance" of X**

  $Var(Y) = Cov(Y,Y) = E\left(Y^2\right) - \left[E(Y)\right]^2 =$ **"Variance" of Y**

- $Corr(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} =$ **"Correlation" of X and Y =** r_{XY}

  $Var(X) = s_X^2,\ \ Var(Y) = s_Y^2,$

  $Cov(X,Y) = r_{XY}\, s_X\, s_Y$

Variance of a Sum

- $Var(X+Y) = E\left([X+Y]^2\right) - [E(X+Y)]^2$

$$= E\left(X^2 + 2XY + Y^2\right) - [E(X) + E(Y)]^2$$

$$= E\left(X^2\right) + 2E(XY) + E(Y)^2 - [E(X)]^2 - 2E(X)E(Y) - [E(Y)]^2$$

$$= E\left(X^2\right) - [E(X)]^2 + E\left(Y^2\right) - [E(Y)]^2 + 2[E(XY) - E(X)E(Y)]$$

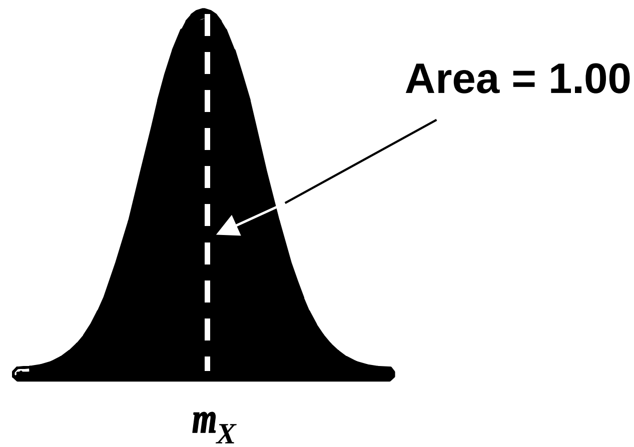
$$= Var(X) + Var(Y) + 2Cov(X,Y)$$

$$= Var(X) + Var(Y) + 2Corr(X,Y)\sqrt{Var(X)Var(Y)}$$

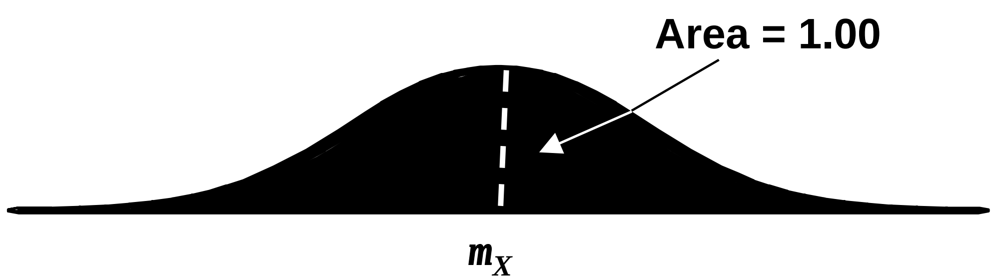
$$= s_X^2 + s_Y^2 + 2r_{XY}s_Xs_Y$$

Variance Measures Uncertainty

- s_X^2 Small



- s_X^2 Large



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Correlation Affects the Variance

- $X_1,\ X_2,\ \ldots,\ X_n$ **are Costs of WBS Elements (Random Variables)**
- **Total Cost =** $\sum_{k=1}^{n} X_k = X_1 + X_2 + \ldots + X_n$
- **Mean of Total Cost =** $E\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} E(X_k) = \sum_{k=1}^{n} m_k$
- **Variance of Total Cost =** $Var\left(\sum_{k=1}^{n} X_k\right)$

$$= \sum_{k=1}^{n} s_k^2 + 2\sum_{j=2}^{n} \sum_{i=1}^{j-1} r_{ij} s_i s_j$$

Does Correlation Matter?

- **If WBS-Element Costs are Uncorrelated (all $\rho_{ij} = 0$),**

  Variance of Total Cost = $\sum_{k=1}^{n} \sigma_k^2$

- **If WBS-Element Costs are Correlated,**

  Variance of Total Cost = $\sum_{k=1}^{n} \sigma_k^2 + 2\sum_{j=2}^{n} \sum_{i=1}^{j-1} \rho_{ij} \sigma_i \sigma_j$

  - **Positive Correlations Increase Dispersion**
  - **Negative Correlations Reduce Dispersion**

- **If (“Worst” Case) All Correlations $\rho_{ij} = 1$,**

  Variance of Total Cost = $\left(\sum_{k=1}^{n} \sigma_k \right)^2 >> \sum_{k=1}^{n} \sigma_k^2$

- **“Ignoring” Correlation Issue is Tantamount to Setting all $\rho_{ij} = 0$**

Yes, Correlation Matters

- **Suppose for Simplicity**
  - **There are *n* Cost Elements** $C_1,\ C_2,\ \ldots,\ C_n$
  - **Each** $Var(C_i) = s^2$
  - **Each** $Corr(C_i, C_j) = r < 1$
  - **Total Cost** $C = \sum_{k=1}^{n} C_i$
- $Var(C) = \sum_{k=1}^{n} Var(C_i) + 2r \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sqrt{Var(C_i)\ Var(C_j)}$

$$= ns^2 + n(n-1)rs^2$$

$$= ns^2\left(1 + (n-1)r\right)$$

Correlation	0	ρ	1
$Var(C)$	ns^2	$ns^2\left(1+(n-1)r\right)$	n^2s^2

Magnitude of Correlation Impact

- **Percent Underestimation of Total-Cost Sigma $\left(\sqrt{Var(C)}\right)$ When Correlation Assumed to be 0 instead of ρ is 100% times ...**

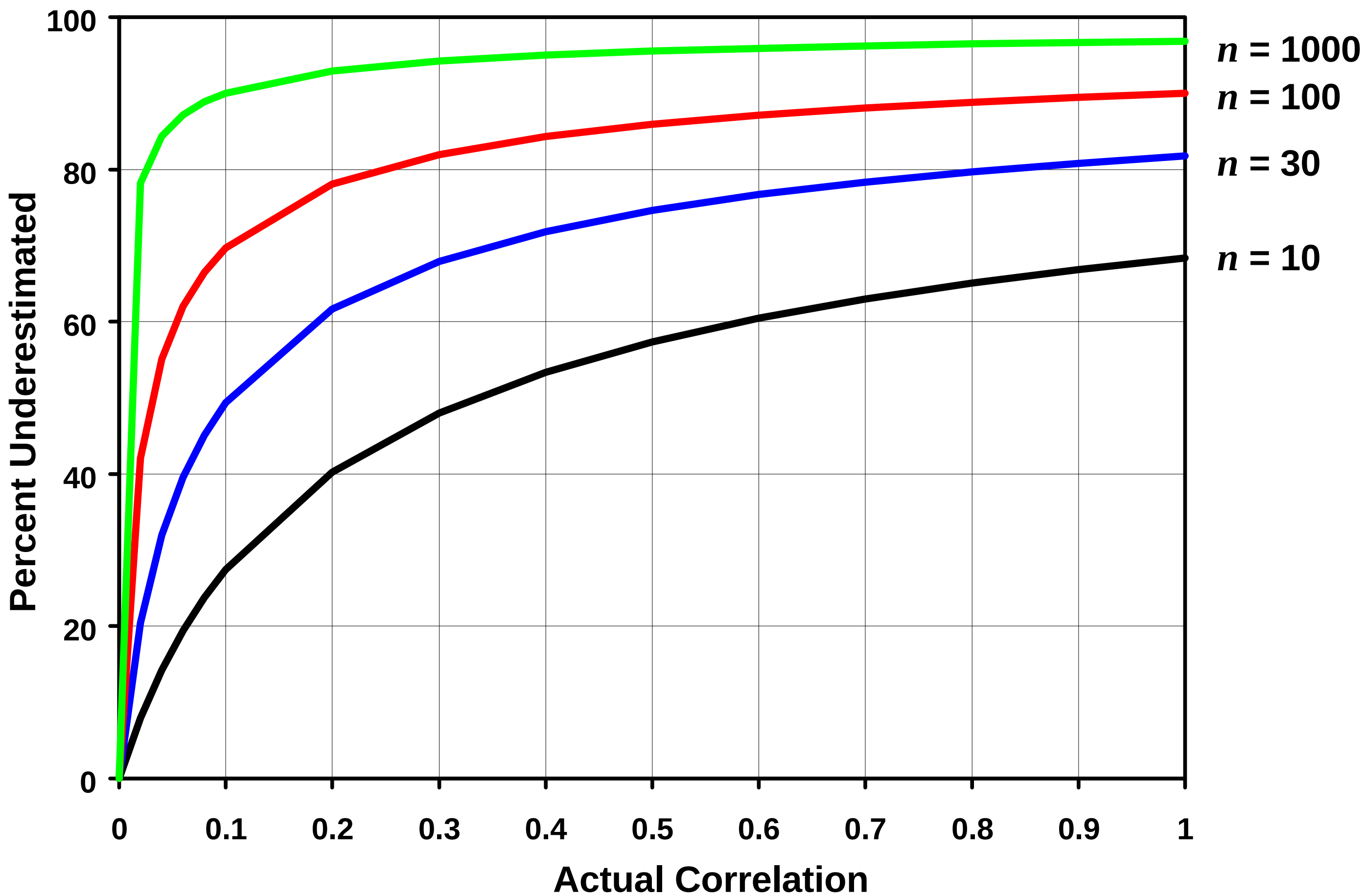
$$\frac{\sqrt{ns}\sqrt{1+(n-1)r}-\sqrt{ns}}{\sqrt{ns}\sqrt{1+(n-1)r}}=1-\sqrt{\frac{1}{1+(n-1)r}}$$

- **Percent Overestimation of Total-Cost Sigma $\left(\sqrt{Var(C)}\right)$ When Correlation Assumed to be 1 instead of ρ is 100% times ...**

$$\frac{ns-\sqrt{ns}\sqrt{1+(n-1)r}}{\sqrt{ns}\sqrt{1+(n-1)r}}=\sqrt{\frac{n}{1+(n-1)r}}-1$$

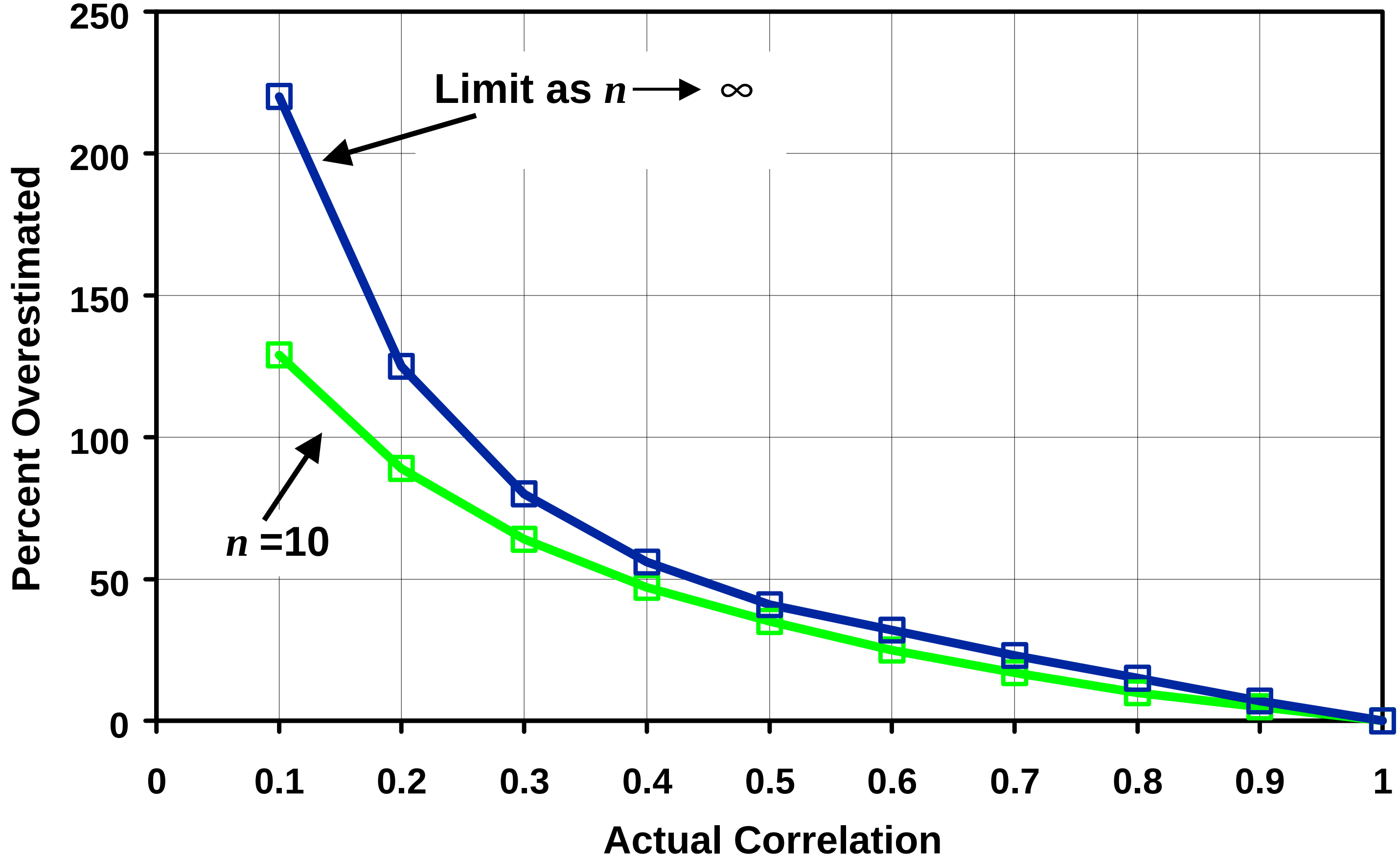
Maximum Possible Underestimation of Total-Cost Sigma

- **Percent Underestimated When Correlation Assumed to be 0 Instead of ρ**



Maximum Possible Overestimation of Total-Cost Sigma

- **Percent Overestimated When Correlation Assumed to be 1 Instead of ρ**



Impact of Nonzero Correlation Value

- **Percent Underestimation of Total-Cost Sigma** $\left(\sqrt{Var(C)}\right)$ **When Correlation Assumed to be** ρ_1 **instead of** $\rho_2 > \rho_1$ **is 100% times ...**

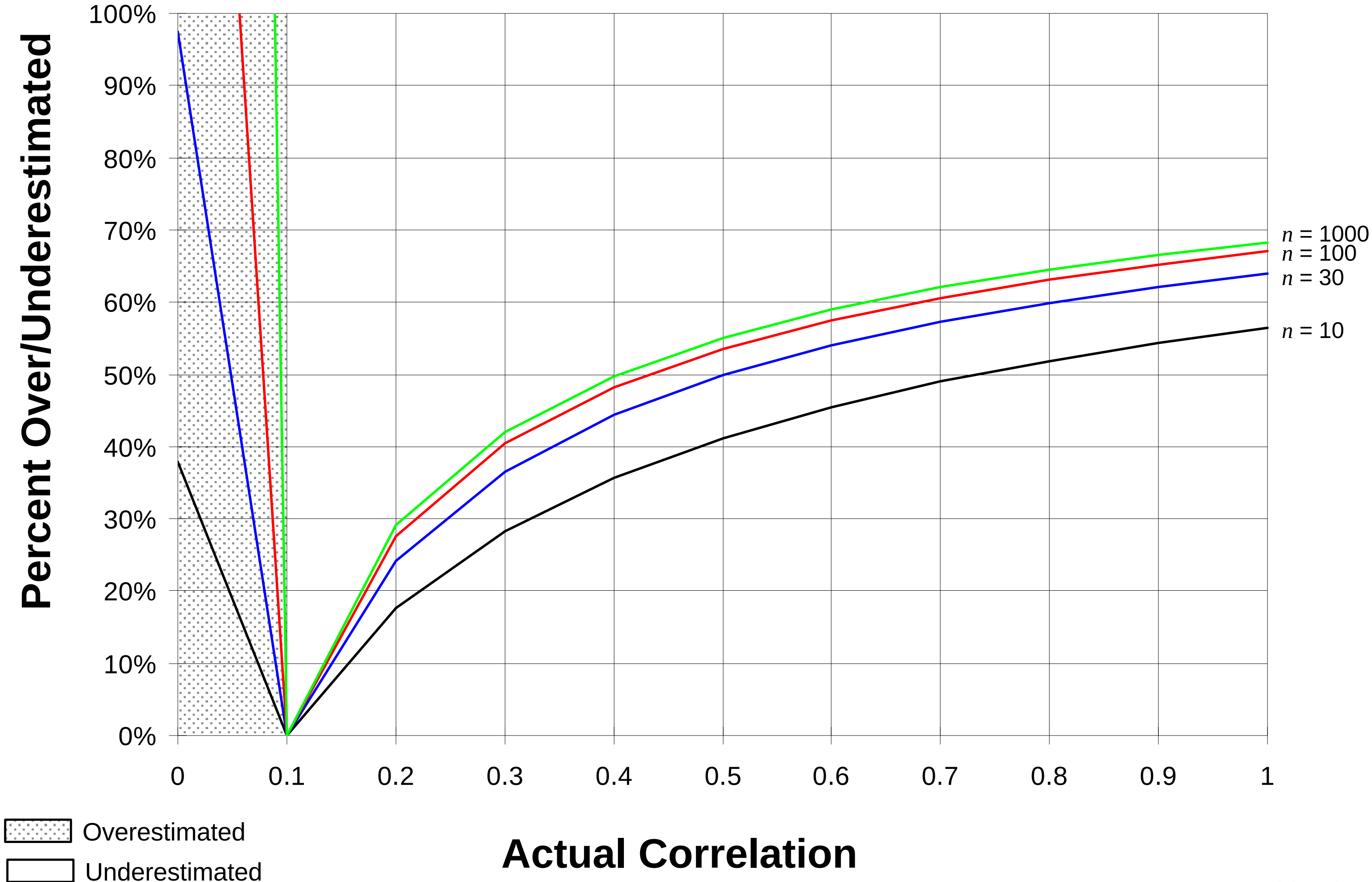
$$1-\sqrt{\frac{1+(n-1)\mathbf{r}_1}{1+(n-1)\mathbf{r}_2}}$$

- **Percent Overestimation of Total-Cost Sigma** $\left(\sqrt{Var(C)}\right)$ **When Correlation Assumed to be** ρ_2 **instead of** $\rho_1 < \rho_2$ **is 100% times ...**

$$\sqrt{\frac{1+(n-1)\mathbf{r}_2}{1+(n-1)\mathbf{r}_1}}-1$$

Maximum Possible Over- and Under-Estimation of Total-Cost Sigma

- **Percent Over/Underestimated When Correlation Assumed to be 0.1 Instead of ρ**



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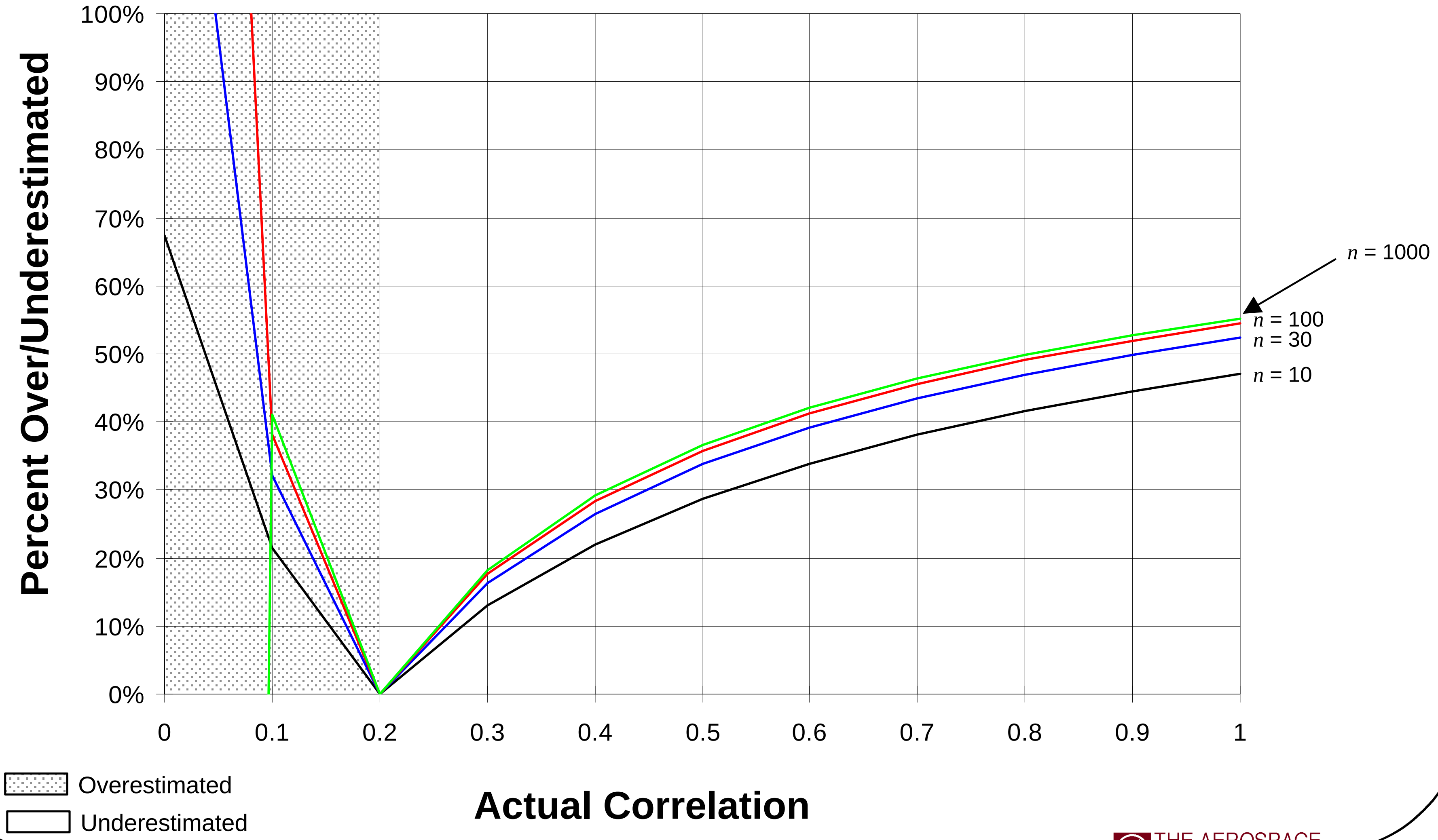
Maximum Possible Over- and Under-Estimation of Total-Cost Sigma

- **Percent Over/Underestimated When Correlation Assumed to be 0.2 Instead of ρ**



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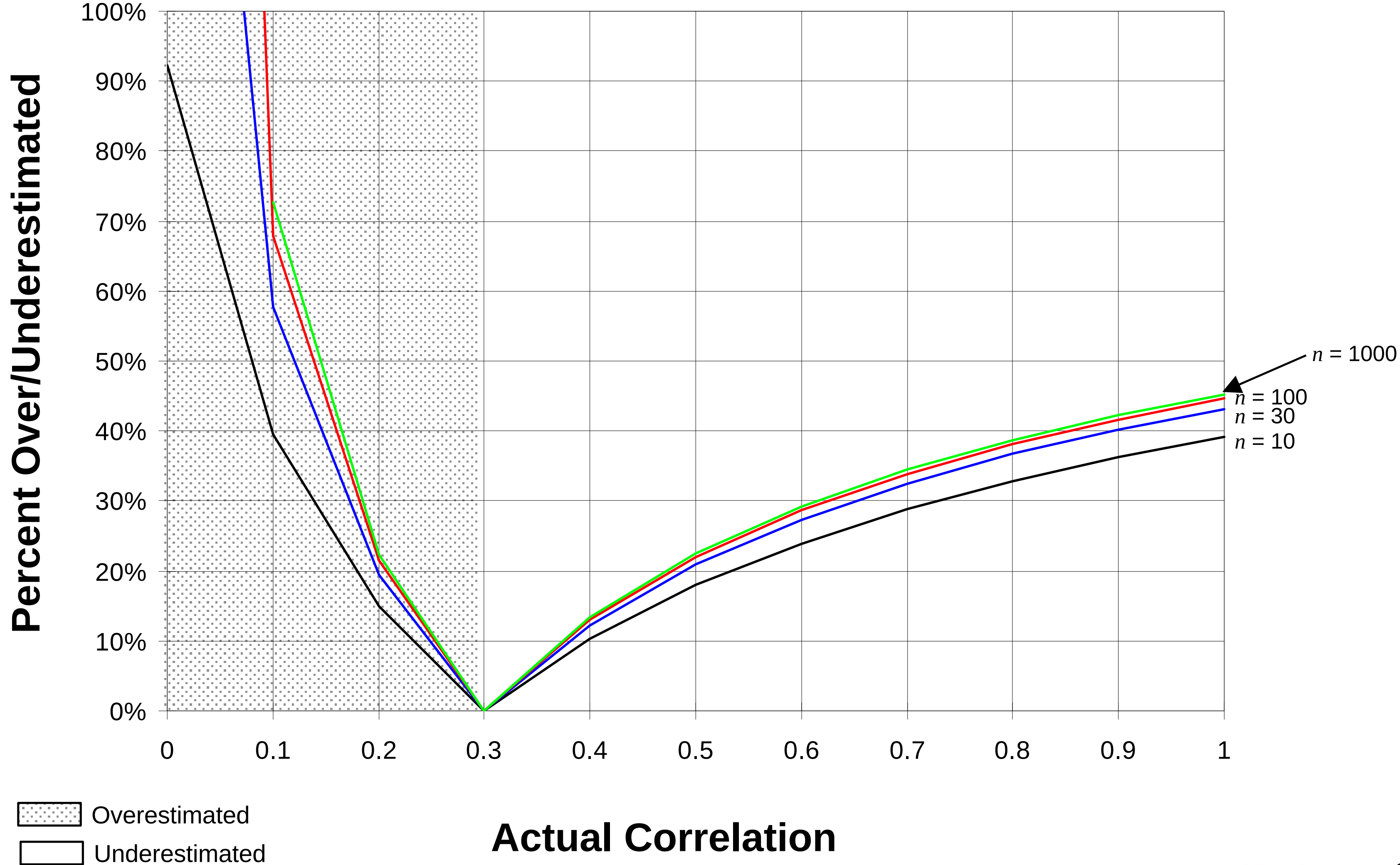
Maximum Possible Over- and Under-Estimation of Total-Cost Sigma

- **Percent Over/Underestimated When Correlation Assumed to be 0.3 Instead of ρ**



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Selection of Correlation Values

- "Ignoring" Correlation is Equivalent to Assuming that Risks are Uncorrelated, e.g., All Correlations are Zero
- Reasonable Choice of Nonzero Values Brings You Closer to Truth
  - 0.2 is at "Knee" of Curve on Previous Charts, thereby Providing Most of the Benefits at Least Loss of Accuracy
- Square of Correlation Represents Percentage of Variation in one WBS Element's Cost Attributable to Influence of Another's

Correlation	% Influenced
0.00	0%
±0.10	1%
±0.32	10%
±0.50	25%
±0.71	50%

A Technical Problem

- **If WBS-Element Costs Are Correlated, Usual Monte Carlo Procedure (Summing Random Numbers Representing WBS-Element Costs) Does Not Produce Total Cost since Random Numbers are Independent**
- **Ideal Way to Simulate Total Cost Would be to Specify Inter-WBS-Element Correlations, Generate Correlated Random Numbers, and Sum Them**
- **Probability Theory Allows This to be Done *Exactly* <u>only</u> for Gaussian Cost Distributions, Not in General**
  - **Not Obvious How to Generate Correlated Random Numbers that Correctly Model WBS-Element Cost Distributions**
  - **Combining Process Must Provide <u>Approximately</u> Correct Values of Mean, Sigma, Percentiles of Total-Cost Distribution**
  - **Drs. M.S. Goldberg and P.M. Lurie of IDA (Institute for Defense Analyses) have been Working on this Problem for Several Years**

There is a Second Type of Correlation

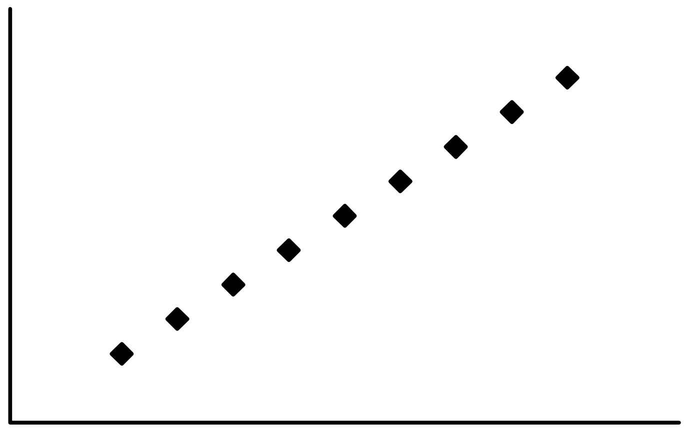
- **Pearson Product-Moment Linear Correlation** $\boldsymbol{r}(X,Y)$
  - $\boldsymbol{r}(X,Y)=\pm 1$ **if and only if X and Y are linearly related, i.e., the least-squares linear relationship between X and Y allows us to predict Y precisely, given X**
  - $\boldsymbol{r}^2(X,Y)$ **= proportion of variation in Y that can be explained on the basis of a least-squares linear relationship between X and Y**
  - $\boldsymbol{r}(X,Y)=0$ **if and only if the least-squares linear relationship between X and Y provides no ability to predict Y, given X**

- **Spearman Rank Correlation** $\boldsymbol{r}_s(X,Y)$
  - $\boldsymbol{r}_s(X,Y)=1$ **if and only if the largest value of X corresponds to the largest value of Y , the second largest, ... , etc.**
  - $\boldsymbol{r}_s(X,Y)=-1$ **if and only if the largest value of X corresponds to the smallest value of Y, etc.**
  - $\boldsymbol{r}_s(X,Y)=0$ **if and only if the rank of a particular X among all X values provides no ability to predict the rank of the corresponding Y among all Y values**

How Do The Two Types of Correlation Differ?

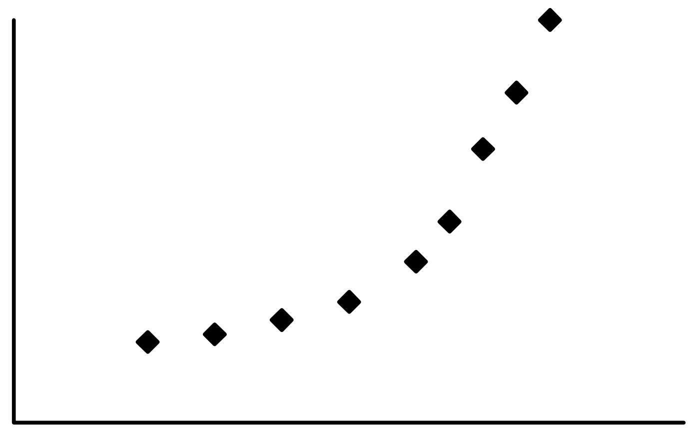
- <u>Pearson</u> Correlation
  - Measures Extent of <u>Linearity</u> of a Relationship Between Two Random Variables
  - Plays an Explicit, Well-defined Role in Establishing the Sigma Value (as well as the Range) of the Total-Cost Distribution
- <u>Spearman</u> Correlation
  - Measures Extent of <u>Monotonicity</u> of a Relationship Between Two Random Variables
  - Does Not Appear Explicitly in the Formula for the Total-Cost Sigma (Its impact on Sigma is Not Known)
- P.R. Garvey of The MITRE Corporation has Done Research on the Issue of Pearson vs. Spearman Correlation

Linear vs. Rank Correlation



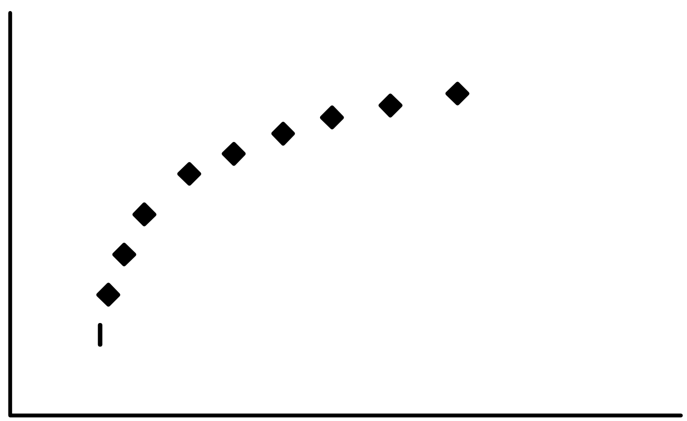
$r = 1.0$

$r_s = 1.0$



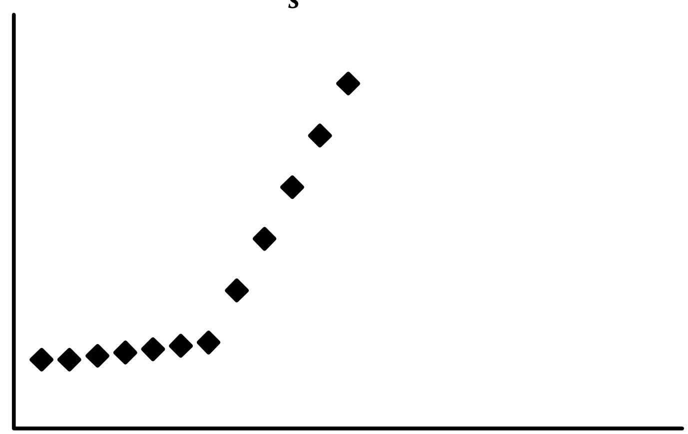
$r = 0.8$

$r_s = 1.0$



$r = 0.8$

$r_s = 1.0$



$r = 0.6$

$r_s = 1.0$

Spearman Rank Correlation Coefficient

- **Data Structure**

CASE	RANK OF X_i VALUE	RANK OF X_j VALUE	DIFFERENCE	SQUARED DIFFERENCE
#1	r_1	c_1	$d_1 = c_1 - r_1$	d_1^2
#2	r_2	c_2	$d_2 = c_2 - r_2$	d_2^2
#3	r_3	c_3	$d_3 = c_3 - r_3$	d_3^2
⋮	⋮	⋮	⋮ ⋮	⋮
#4	r_n	c_n	$d_n = c_n - r_n$	d_n^2
SUMS	$\frac{n(n+1)}{2}$	$\frac{n(n+1)}{2}$	$\sum d = \sum c - \sum r = 0$	$\sum d^2$

$$r\left(X_i, X_j\right) = 1 - \frac{6\sum d^2}{n\left(n^2 - 1\right)}$$

- **Statistics Theorem: Spearman Rank Correlation Coefficient Equals Pearson (Linear) Correlation Coefficient Calculated Between the Two Sets of Ranks**

Crystal Ball, @Risk

- **Commercially Available Software Packages that are Add-ons to Additional Commercial Software Such As Windows, Excel, or Lotus on PC or Mac**
  - **Crystal Ball Marketed by Decisioneering, Inc., 2530 S. Parker Road, Suite 220, Aurora, CO 80014, (800) 289-2550**
  - **@Risk Marketed by Palisade Corporation, 31 Decker Road, Newfield, NY 14867, (800) 432-7475**
- **Inputs**
  - **Parameters Defining WBS-Element Distributions**
  - **Rank Correlations Among WBS-Element Cost Distributions**
- **Mathematics**
  - **Monte-Carlo and Stratified Random Sampling (Latin Hypercube)**
  - **Virtually All Probability Distributions That Have Names Can Be Used**
  - **Suggests Adjustments to Inconsistent Input Correlation Matrix**
- **Outputs**
  - **Percentiles of Program Cost**
  - **Cost Probability Density and Cumulative Distribution Graphics**

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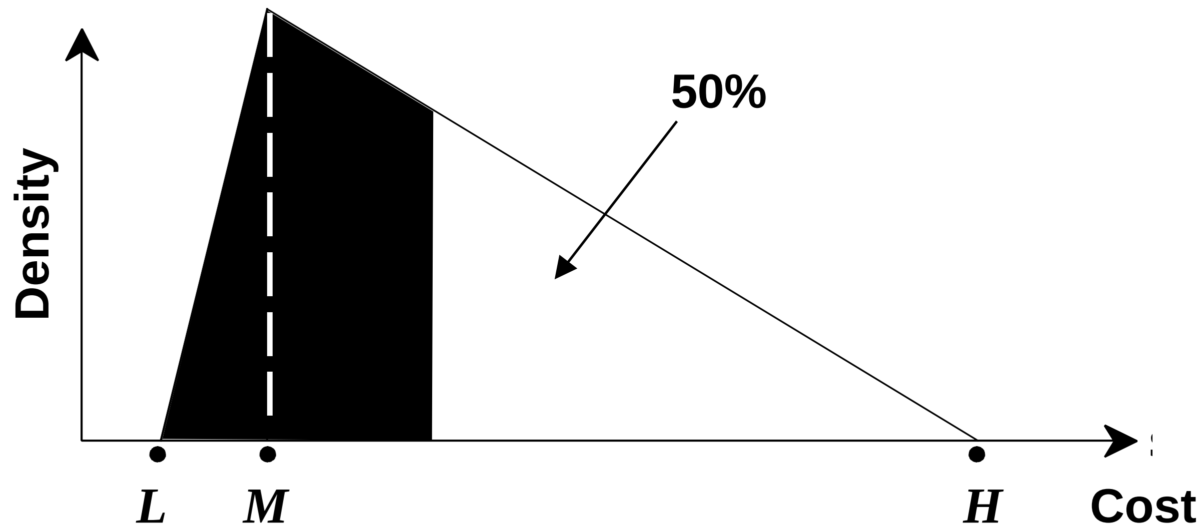
Example: Triangular Cost Distribution



- **Probability Density Function**
- **Total Area = 1.00**
- **Three Parameters *L*, *M*, *H* Completely Specify Distribution**
- **Mean, Median, Sigma, All Percentiles can be Expressed in Terms of *L*, *M*, and *H***

Statistical Metrics of Triangular Distribution

- **Mode = M (most likely value of cost)**
- **Median =** $T_{.50} = L + \sqrt{0.50(M-L)(H-L)}$ if $M - L \geq 0.50(H-L)$

  $= H - \sqrt{0.50(H-L)(H-M)}$ if $M - L \leq 0.50(H-L)$

- T_p **= Dollar Value at Which** $P\{\text{Cost} \leq T_p\} = p$

  $T_p = H - \sqrt{(1-p)(H-L)(H-M)}$ if $p \geq \dfrac{M-L}{H-L}$

  $T_p = L + \sqrt{p(M-L)(H-L)}$ if $p \leq \dfrac{M-L}{H-L}$

- **Mean =** $\dfrac{L+M+H}{3}$

$$s = \sqrt{\frac{L^2 + M^2 + H^2 - LM - LH - MH}{18}}$$

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Sum of Five Uncorrelated Triangular Distributions

- **Monte-Carlo Simulation Using @Risk™ Software**
- **Input Parameters: L = 100, M = 200, H = 500**
- **Comparison of Rolled-Up vs. Correct Total-Cost Statistics**

STATISTIC	EACH INPUT DISTRIBUTION	ROLLED-UP VALUE	CORRECT VALUE	OVER-ESTIMATE: ROLL-UP, MINUS CORRECT
MEAN	266.67	1333.33	1333.33	0.00
MODE	200.00	1000.00	1236.94	-236.94
MEDIAN	255.05	1275.25	1328.53	-53.28
STANDARD DEVIATION	84.98	190.03	190.03	0.00
50th PERCENTILE	255.05	1275.25	1328.53	-53.28
70th PERCENTILE	310.25	1551.30	1432.52	118.78
90th PERCENTILE	390.46	1952.30	1573.66	378.64

Sum of Five Correlated Triangular Distributions

- **Monte-Carlo Simulation Using @Risk™ Software**
- **Input Parameters: L = 100, M = 200, H = 500**
- **All Pairwise Correlations are 0.10**
- **Comparison of Rolled-Up vs. Correct Total-Cost Statistics**

STATISTIC	EACH INPUT DISTRIBUTION	ROLLED-UP VALUE	CORRECT VALUE	OVER-ESTIMATE: ROLL-UP, MINUS CORRECT
MEAN	266.67	1333.33	1333.33	0.00
MODE	200.00	1000.00	1192.82	-192.82
MEDIAN	255.05	1275.25	1319.20	-43.95
STANDARD DEVIATION	84.98	190.03	224.85	-34.82
50th PERCENTILE	255.05	1275.25	1319.20	-43.95
70th PERCENTILE	310.25	1551.30	1444.29	107.01
90th PERCENTILE	390.46	1952.30	1635.07	317.23

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Sum of Five Correlated Triangular Distributions

- **Monte-Carlo Simulation Using @Risk™ Software**
- **Input Parameters: L = 100, M = 200, H = 500**
- **All Pairwise Correlations are 0.30**
- **Comparison of Rolled-Up vs. Correct Total-Cost Statistics**

STATISTIC	EACH INPUT DISTRIBUTION	ROLLED-UP VALUE	CORRECT VALUE	OVER-ESTIMATE: ROLL-UP, MINUS CORRECT
MEAN	266.67	1333.33	1333.33	0.00
MODE	200.00	1000.00	1187.18	-187.18
MEDIAN	255.05	1275.25	1305.28	-30.03
STANDARD DEVIATION	84.98	190.03	281.86	-91.83
50th PERCENTILE	255.05	1275.25	1305.28	-30.03
70th PERCENTILE	310.25	1551.30	1469.59	81.71
90th PERCENTILE	390.46	1952.30	1715.72	236.58

Sum of Five Correlated Triangular Distributions

- **Monte-Carlo Simulation Using @Risk™ Software**
- **Input Parameters: L = 100, M = 200, H = 500**
- **All Pairwise Correlations are 0.50**
- **Comparison of Rolled-Up vs. Correct Total-Cost Statistics**

STATISTIC	EACH INPUT DISTRIBUTION	ROLLED-UP VALUE	CORRECT VALUE	OVER-ESTIMATE: ROLL-UP, MINUS CORRECT
MEAN	266.67	1333.33	1333.33	0.00
MODE	200.00	1000.00	1181.76	-181.76
MEDIAN	255.05	1275.25	1297.63	-22.38
STANDARD DEVIATION	84.98	190.03	329.14	-139.11
50th PERCENTILE	255.05	1275.25	1297.63	-22.38
70th PERCENTILE	310.25	1551.30	1494.05	57.25
90th PERCENTILE	390.46	1952.30	1777.48	174.82

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Sum of Five Correlated Triangular Distributions

- Monte-Carlo Simulation Using @Risk™ Software
- Input Parameters: L = 100, M = 200, H = 500
- All Pairwise Correlations are 0.80
- Comparison of Rolled-Up vs. Correct Total-Cost Statistics

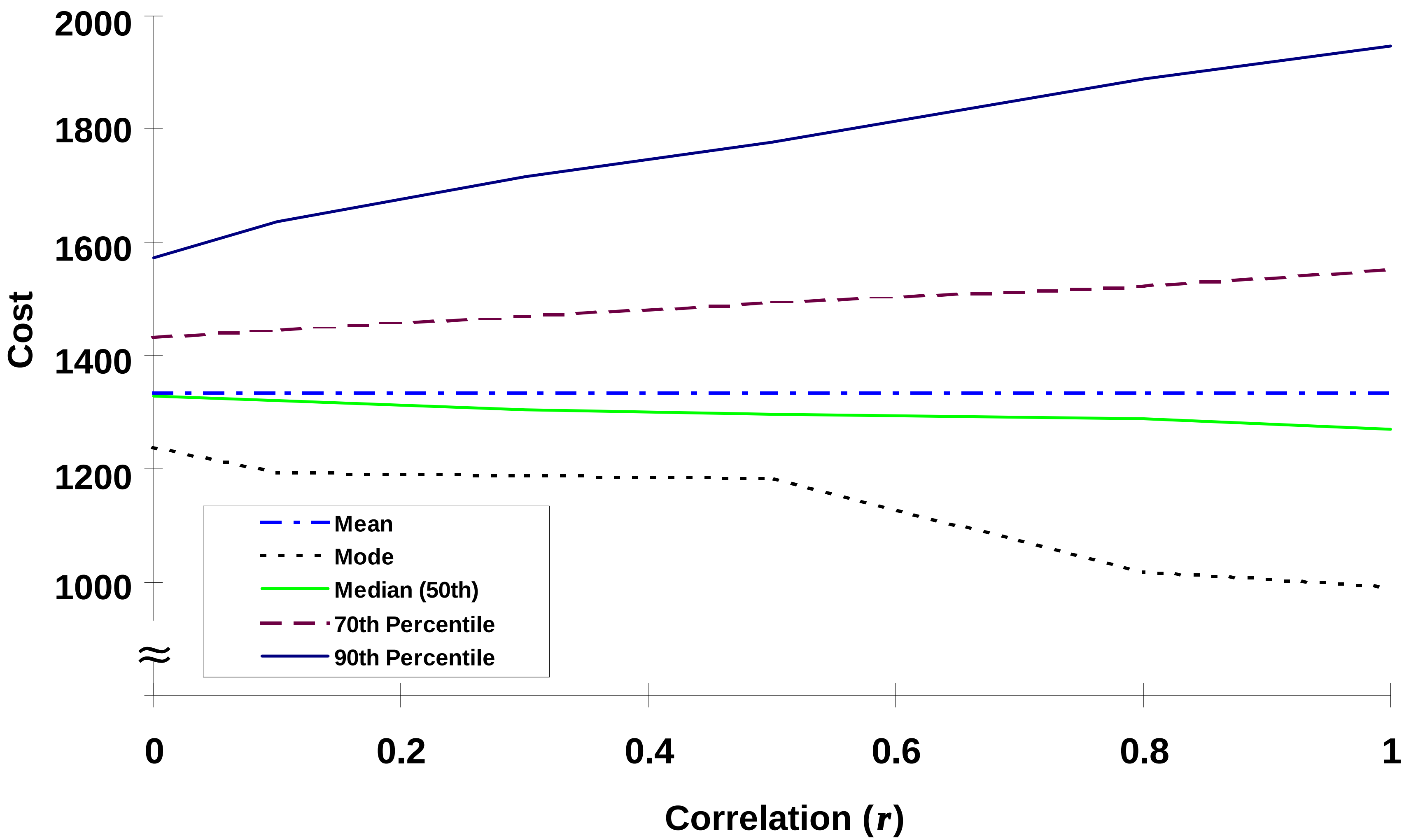
STATISTIC	EACH INPUT DISTRIBUTION	ROLLED-UP VALUE	CORRECT VALUE	OVER-ESTIMATE: ROLL-UP, MINUS CORRECT
MEAN	266.67	1333.33	1333.33	0.00
MODE	200.00	1000.00	1018.70	-18.70
MEDIAN	255.05	1275.25	1289.00	-13.75
STANDARD DEVIATION	84.98	190.03	389.44	-199.41
50th PERCENTILE	255.05	1275.25	1289.00	-13.75
70th PERCENTILE	310.25	1551.30	1522.95	28.35
90th PERCENTILE	390.46	1952.30	1887.31	64.99

Sum of Five Correlated Triangular Distributions

- **Monte-Carlo Simulation Using @Risk™ Software**
- **Input Parameters: L = 100, M = 200, H = 500**
- **All Pairwise Correlations are 1.00**
- **Comparison of Rolled-Up vs. Correct Total-Cost Statistics**

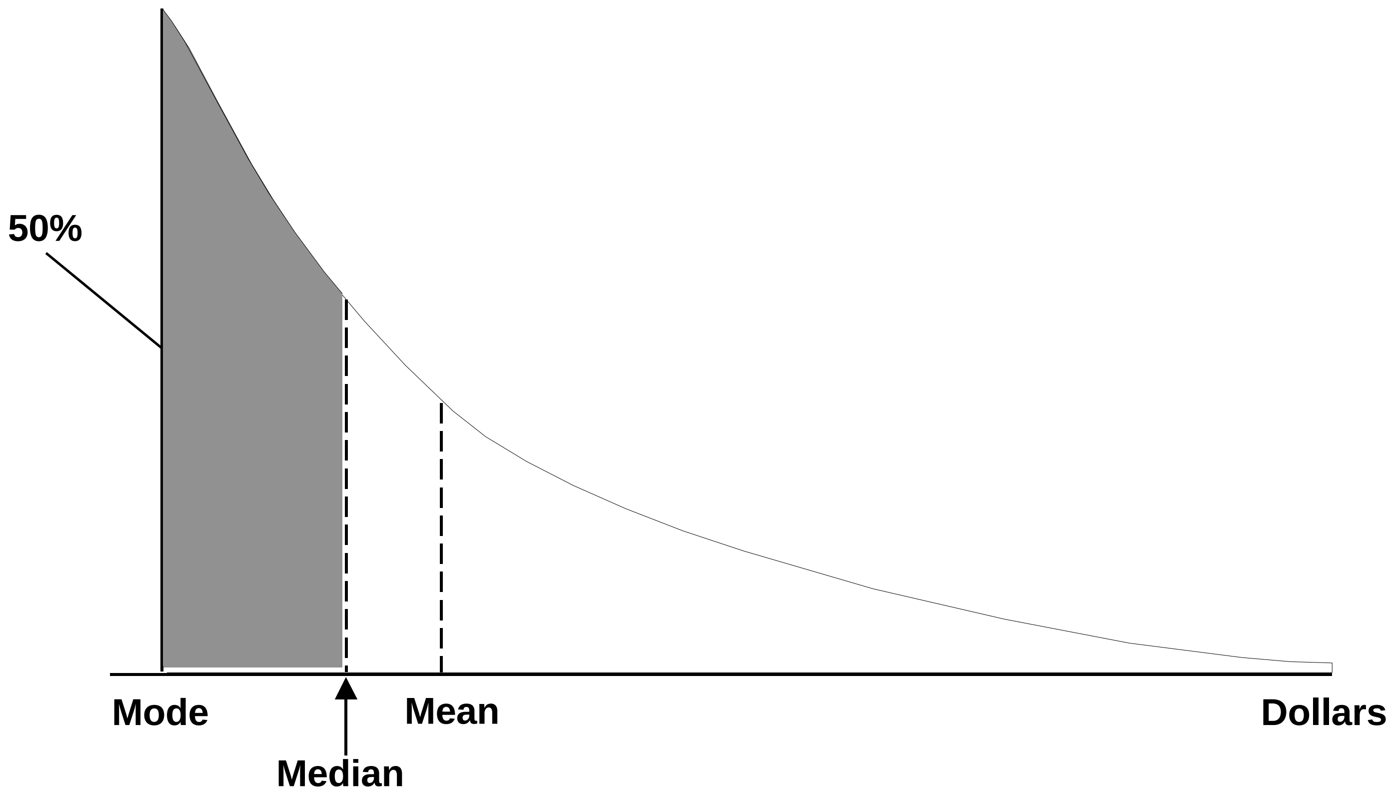
STATISTIC	EACH INPUT DISTRIBUTION	ROLLED-UP VALUE	CORRECT VALUE	OVER-ESTIMATE: ROLL-UP, MINUS CORRECT
MEAN	266.67	1333.33	1333.33	0.00
MODE	200.00	1000.00	990.69	9.31
MEDIAN	255.05	1275.25	1271.09	4.16
STANDARD DEVIATION	84.98	190.03	424.92	-234.89
50th PERCENTILE	255.05	1275.25	1271.09	4.16
70th PERCENTILE	310.25	1551.30	1550.07	1.23
90th PERCENTILE	390.46	1952.30	1946.27	6.03

Effects of Increasing Correlation on the Sum of Five Triangular Distributions

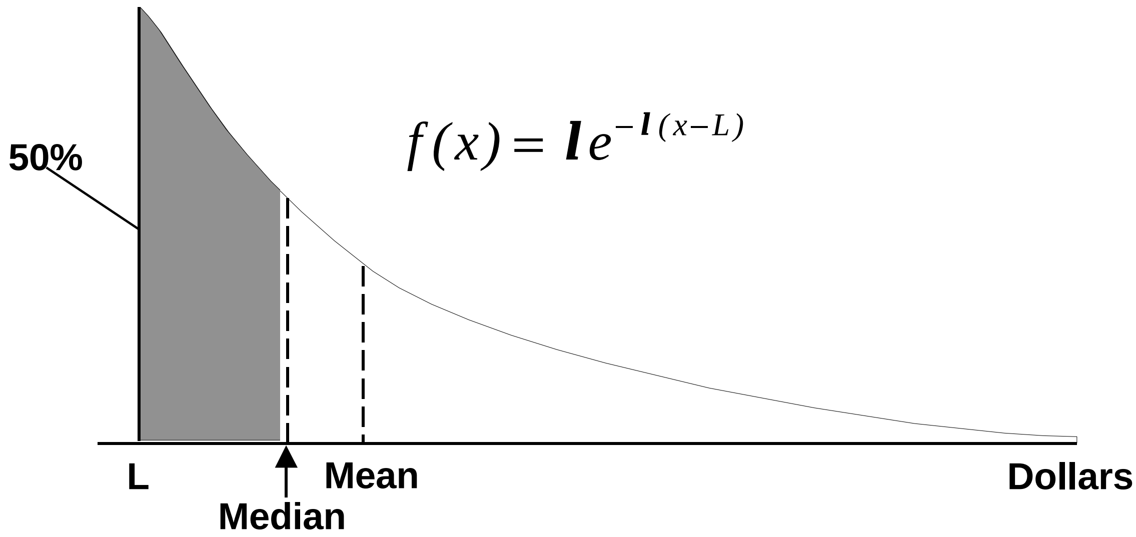


Example : Exponential Distribution

- **Five WBS Elements**
- **WBS-Element Cost Distributions Each Have Exponential Distribution**



Exponential Cost Distribution



$$f(x) = \lambda e^{-\lambda(x-L)}$$

- *f(x)* **is the Exponential Probability Density Function**
- **Total Area = 1.00**
- **Two Parameters L, λ Completely Specify Distribution**
- **Mean, Median, Sigma, All Percentiles can be Expressed in Terms of L and λ**

Statistical Metrics of Exponential Distribution

- **Mode = L**

- **Median = 50th Percentile =** $L + \dfrac{\ln 2}{l}$

- E_p **= Dollar Value at Which** $P\{\text{Cost} \leq E_p\}$ **is** p

$$E_p = L - \frac{\ln(1-p)}{l}$$

- **Mean =** $L + \dfrac{1}{l}$

- $Q = \dfrac{1}{l}$

Sum of Five Uncorrelated Exponential Distributions

- **Monte-Carlo Simulation Using @Risk™ Software**
- **Input Parameters: L = 100, M = 200, H = 500**
- **Comparison of Rolled-Up vs. Correct Total-Cost Statistics**

STATISTIC	EACH INPUT DISTRIBUTION	ROLLED-UP VALUE	CORRECT VALUE	OVER-ESTIMATE: ROLL-UP, MINUS CORRECT
MEAN	266.67	1333.33	1333.33	0.00
MODE	100.00	500.00	1153.04	-653.04
MEDIAN	215.52	1077.62	1287.83	-210.21
STANDARD DEVIATION	166.67	372.68	372.68	0.00
50th PERCENTILE	215.52	1077.62	1287.83	-210.21
70th PERCENTILE	300.66	1503.31	1486.16	17.15
90th PERCENTILE	483.76	2418.82	1809.35	609.47

Sum of Five Correlated Exponential Distributions

- **Monte-Carlo Simulation Using @Risk™ Software**
- **Input Parameters: L = 100, M = 200, H = 500**
- **All Pairwise Correlations are 0.10**
- **Comparison of Rolled-Up vs. Correct Total-Cost Statistics**

STATISTIC	EACH INPUT DISTRIBUTION	ROLLED-UP VALUE	CORRECT VALUE	OVER-ESTIMATE: ROLL-UP, MINUS CORRECT
MEAN	266.67	1333.33	1333.33	0.00
MODE	100.00	500.00	1027.80	-527.80
MEDIAN	215.52	1077.62	1249.68	-172.06
STANDARD DEVIATION	166.67	372.68	440.96	-68.28
50th PERCENTILE	215.52	1077.62	1249.68	-172.06
70th PERCENTILE	300.66	1503.31	1481.14	22.17
90th PERCENTILE	483.76	2418.82	1903.46	515.36

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Sum of Five Correlated Exponential Distributions

- **Monte-Carlo Simulation Using @Risk™ Software**
- **Input Parameters: L = 100, M = 200, H = 500**
- **All Pairwise Correlations are 0.30**
- **Comparison of Rolled-Up vs. Correct Total-Cost Statistics**

STATISTIC	EACH INPUT DISTRIBUTION	ROLLED-UP VALUE	CORRECT VALUE	OVER-ESTIMATE: ROLL-UP, MINUS CORRECT
MEAN	266.67	1333.33	1333.33	0.00
MODE	100.00	500.00	981.76	-481.76
MEDIAN	215.52	1077.62	1205.95	-128.33
STANDARD DEVIATION	166.67	372.68	552.77	-180.09
50th PERCENTILE	215.52	1077.62	1205.95	-128.33
70th PERCENTILE	300.66	1503.31	1498.84	4.47
90th PERCENTILE	483.76	2418.82	2053.84	-364.98

Sum of Five Correlated Exponential Distributions

- **Monte-Carlo Simulation Using @Risk™ Software**
- **Input Parameters: L = 100, M = 200, H = 500**
- **All Pairwise Correlations are 0.50**
- **Comparison of Rolled-Up vs. Correct Total-Cost Statistics**

STATISTIC	EACH INPUT DISTRIBUTION	ROLLED-UP VALUE	CORRECT VALUE	OVER-ESTIMATE: ROLL-UP, MINUS CORRECT
MEAN	266.67	1333.33	1333.33	0.00
MODE	100.00	500.00	767.35	-267.35
MEDIAN	215.52	1077.62	1165.85	-88.23
STANDARD DEVIATION	166.67	372.68	645.50	-272.82
50th PERCENTILE	215.52	1077.62	1165.85	-88.23
70th PERCENTILE	300.66	1503.31	1497.09	6.22
90th PERCENTILE	483.76	2418.82	2155.59	263.23

Sum of Five Correlated Exponential Distributions

- **Monte-Carlo Simulation Using @Risk™ Software**
- **Input Parameters: L = 100, M = 200, H = 500**
- **All Pairwise Correlations are 0.80**
- **Comparison of Rolled-Up vs. Correct Total-Cost Statistics**

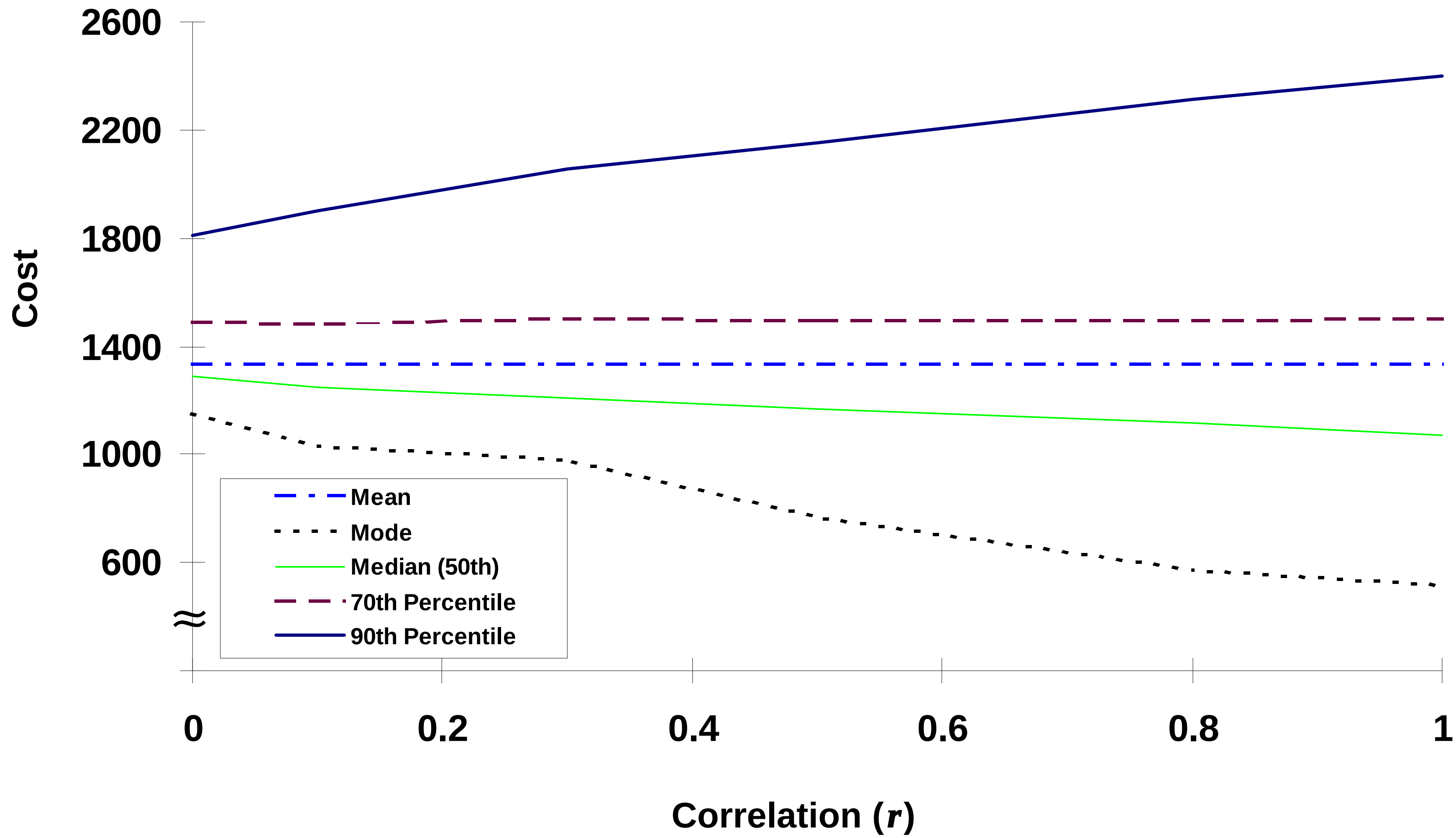
STATISTIC	EACH INPUT DISTRIBUTION	ROLLED-UP VALUE	CORRECT VALUE	OVER-ESTIMATE: ROLL-UP, MINUS CORRECT
MEAN	266.67	1333.33	1333.33	0.00
MODE	100.00	500.00	574.59	-74.59
MEDIAN	215.52	1077.62	1119.08	-41.46
STANDARD DEVIATION	166.67	372.68	763.76	-391.08
50th PERCENTILE	215.52	1077.62	1119.08	-41.46
70th PERCENTILE	300.66	1503.31	1492.81	10.50
90th PERCENTILE	483.76	2418.82	2311.91	106.91

Sum of Five Correlated Exponential Distributions

- **Monte-Carlo Simulation Using @Risk™ Software**
- **Input Parameters: L = 100, M = 200, H = 500**
- **All Pairwise Correlations are 1.00**
- **Comparison of Rolled-Up vs. Correct Total-Cost Statistics**

STATISTIC	EACH INPUT DISTRIBUTION	ROLLED-UP VALUE	CORRECT VALUE	OVER-ESTIMATE: ROLL-UP, MINUS CORRECT
MEAN	266.67	1333.33	1333.33	0.00
MODE	100.00	500.00	512.50	-12.50
MEDIAN	215.52	1077.62	1071.97	5.65
STANDARD DEVIATION	166.67	372.68	833.33	-460.65
50th PERCENTILE	215.52	1077.62	1071.97	5.65
70th PERCENTILE	300.66	1503.31	1501.15	2.16
90th PERCENTILE	483.76	2418.82	2400.81	18.01

Effects of Increasing Correlation on the Sum of Five Exponential Distributions



Contents

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Summary

- Every Cost-Analysis Job Requires a Risk Analysis
- WBS-Element Risks(and therefore Costs) are Typically Correlated
- Correlations Impact the Probable Cost Range
- Correlations Also Impact Cost Estimates, namely the Mode, Median, and other Percentiles of the Cost Probability Distribution
- Your Estimate and Range will be Closer to the Truth if You Use Reasonable Nonzero Correlations Rather Than Zeroes